**Revised on 2023-11-14**

# 1 Chapter 2: Regular Surfaces.

**Remark 1.1** *You may have to read the Appendix of this chapter in p.120 first, if you have forgotten some stuff in Advanced Calculus.*

## 1.1 The First Fundamental Form; Area (this is Section 2-5 of the textbook).

Let $S \subset \mathbb{R}^3$ be a regular surface. The natural inner product of $\mathbb{R}^3$ induces on each tangent plane $T_pS \subset \mathbb{R}^3$ an inner product, denoted as $\langle , \rangle_p$. For any $w_1,\ w_2 \in T_pS$, then $\langle w_1, w_2 \rangle_p$ is equal to the inner product of $w_1$ and $w_2$ viewed as vectors in $\mathbb{R}^3$.

**Definition 1.2** *The map*

$$I_p(w) = \langle w, w \rangle_p = |w|^2 : T_pS \to \mathbb{R} \tag{1}$$

*is called the* ***first fundamental form*** *of the regular surface $S$ at $p \in S$. In Linear Algebra terminology, $I_p(w)$ is called a* ***quadratic form*** *on the tangent space $T_pS$.*

**Remark 1.3** *The first fundamental form is the expression of how the surface $S$ inherits the natural inner product of $\mathbb{R}^3$. One can use it to measure the length of curves on $S$ or the area of a region on $S$.*

**Definition 1.4** *If $\mathbf{x}(u,v) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a local parametrization at $p = \mathbf{x}(u_0, v_0) \in S$ with basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, then the following three functions*

$$E(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p, \quad F(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p, \quad G(u_0, v_0) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p, \tag{2}$$

*are called* ***the coefficients of the first fundamental form in the basis*** $\{\mathbf{x}_u, \mathbf{x}_v\}$ ***of*** $T_pS$.

**Remark 1.5** *Since $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ are differentiable map of $(u,v) \in U$ into $\mathbb{R}^3$, the three functions*

$$E(u,v), \quad F(u,v), \quad G(u,v) : (u,v) \in U \to \mathbb{R}$$

*are all differentiable on $U$.*

If $\alpha(t) = \mathbf{x}(u(t), v(t)) : (-\varepsilon, \varepsilon) \to S$ is a differentiable curve with $\alpha(0) = \mathbf{x}(u_0, v_0) = p$, then

$$\begin{aligned} I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0) \rangle_p \\ &= \langle \mathbf{x}_u(u_0,v_0) u'(0) + \mathbf{x}_v(u_0,v_0) v'(0), \mathbf{x}_u(u_0,v_0) u'(0) + \mathbf{x}_v(u_0,v_0) v'(0) \rangle_p \\ &= E(u_0,v_0)(u'(0))^2 + 2F(u_0,v_0) u'(0) v'(0) + G(u_0,v_0)(v'(0))^2 \\ &= \underbrace{(u'(0), v'(0)) \begin{pmatrix} E(u_0,v_0) & F(u_0,v_0) \\ F(u_0,v_0) & G(u_0,v_0) \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}}. \end{aligned}$$

**Lemma 1.6** *The matrix*

$$\begin{pmatrix} E(u_0,v_0) & F(u_0,v_0) \\ F(u_0,v_0) & G(u_0,v_0) \end{pmatrix} \tag{3}$$

*is* ***symmetric*** *and* ***positive definite****.*

**Proof.** It is clear that it is symmetric. To show positive definite, it is equivalent to saying that

$$E(u_0,v_0)>0,\quad E(u_0,v_0)\,G(u_0,v_0)-F^2(u_0,v_0)>0 \tag{4}$$

((4) will imply that $G(u_0,v_0)>0$ also). This is easy by the Cauchy-Schwarz inequality. □

**Remark 1.7** *One can also use definition to see that (3) is positive definite since it is the components of an inner product, which is positive definite, or one can see that*

$$(a,b)\begin{pmatrix} E(u_0,v_0) & F(u_0,v_0)\\ F(u_0,v_0) & G(u_0,v_0)\end{pmatrix}\begin{pmatrix} a\\ b\end{pmatrix}>0,\quad \forall\ (a,b)\neq(0,0)\in\mathbb{R}^2.$$

**Example 1.8** *Do Example 1, 2, 3 in p. 95.*

We can use the first fundamental form $I$ to answer metric questions on $S$ without further references to the ambient space $\mathbb{R}^3$. Let $\alpha(t):0\in I\to S\subset\mathbb{R}^3$ be a curve on $S$. Its **arc length parameter** $s$ is given by

$$\begin{aligned} s(t) &= \int_0^t|\alpha'(t)|\,dt=\int_0^t\sqrt{I(\alpha'(t))}dt,\quad \alpha'(t)=\mathbf{x}_u(u(t),v(t))\,u'(t)+\mathbf{x}_v(u(t),v(t))\,v'(t)\\ &= \int_0^t\sqrt{E(u(t),v(t))\,(u'(t))^2+2F(u(t),v(t))\,u'(t)\,v'(t)+G(u(t),v(t))\,(v'(t))^2}dt\\ &= \int_0^t\sqrt{E(u')^2+2Fu'v'+G(v')^2}dt,\quad t\in I. \end{aligned}\tag{5}$$

**Remark 1.9** *(**Notation.**) We usually write (5) as the convenient form*

$$ds^2=Edu^2+2Fdudv+Gdv^2,$$

*which means that if $\alpha(t)=\mathbf{x}(u(t),v(t))$ is a curve on $S$ with arc length $s=s(t)$, then*

$$\left(\frac{ds}{dt}\right)^2=E(u(t),v(t))\left(\frac{du}{dt}\right)^2+2F(u(t),v(t))\frac{du}{dt}\frac{dv}{dt}+G(u(t),v(t))\left(\frac{dv}{dt}\right)^2.$$

**Remark 1.10** *If $S$ is the **cylinder** in Example 2 in p. 95, we have $E\equiv1,\ F\equiv0,\ G\equiv1$ and then*

$$s(t)=\int_0^t|\alpha'(t)|\,dt=\int_0^t\sqrt{I(\alpha'(t))}dt=\int_0^t\sqrt{(u'(t))^2+(v'(t))^2}dt,\quad t\in(-\varepsilon,\varepsilon),$$

*i.e., the length between **any two points** of $\alpha(t)$ on $S$ is equal to the length of its corresponding two points on $(u(t),v(t))\in U$. In this case, we say the cylinder is **isometric** to the open set in $\mathbb{R}^2$ :*

$$U=\left\{(u,v)\in\mathbb{R}^2:0<u<2\pi,\ -\infty<v<\infty\right\}\subset\mathbb{R}^2.$$

***From the viewpoint of "metric geometry", the cylinder is the same as the plane.*** *However, they have different curvature (more precisely, **different mean curvature**).*

With the help of the first fundamental form $I$ on $S$, we can also discuss the angle between two vectors on the same tangent space. In particular, if $\alpha(t):I\to S$ and $\beta(t):I\to S$ are two curves on $S$ and they intersect at $t=t_0$, their intersection angle $\theta\in[0,\pi]$ between the two curves is defined as

$$\cos\theta=\frac{\langle\alpha'(t_0),\beta'(t_0)\rangle}{|\alpha'(t_0)|\,|\beta'(t_0)|}\in[-1,1].$$

In particular, the angle $\varphi$ between the two coordinate curves $\mathbf{x}(\cdot,v),\ \mathbf{x}(u,\cdot)$ of a parametrization $\mathbf{x}(u,v)$ is given by

$$\cos\varphi=\frac{\langle\mathbf{x}_u,\mathbf{x}_v\rangle}{|\mathbf{x}_u|\,|\mathbf{x}_v|}=\frac{F}{\sqrt{EG}}.$$

Thus the coordinate curves of a parametrization $\mathbf{x}(u,v)$ are **orthogonal** everywhere if and only if $F(u,v)=0$ for all $(u,v)$. Such a parametrization is called an **orthogonal parametrization.**

**Example 1.11** *Do Example 4 in p. 98.*

**Remark 1.12** *The following integral formula is for your reference when you read p. 99:*

$$\int \frac{1}{\sin\theta} d\theta = \int \csc\theta d\theta = \int \frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta - \cot\theta} d\theta = \int \frac{\frac{d}{d\theta}(\csc\theta - \cot\theta)}{\csc\theta - \cot\theta} d\theta$$
$$= \log|\csc\theta - \cot\theta| = \log\left|\frac{1-\cos\theta}{\sin\theta}\right| = \log\left|\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right| = \log\left|\tan\frac{\theta}{2}\right|.$$

### 1.1.1 Area Formula on a Regular Surface (this is Section 2-5 of the textbook).

**Definition 1.13** *See p. 99 for the definition of a (regular) domain and region on* $S$.

**Remark 1.14** *Explain the meaning of domain and region on* $S$.

Recall that if $a,\ b$ are two vectors in $\mathbb{R}^3$, then

$$|a \wedge b| = \text{area of the parallelogram generated by } a,\ b.$$

Motivated by this, if $\mathbf{x} : U \to S$ is a local parametrization and $Q \subset U$ is a compact region with boundary a piecewise smooth simple closed curve, **we define the area of** $R = \mathbf{x}(Q) \subset S$ **as:**

$$A(R) = \iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v|\, dudv. \tag{6}$$

We also use $\iint_R d\sigma$ to denote the area $A(R)$ of $R \subset S$.

**Lemma 1.15** *The above definition* ***does not*** *depend on the parametrization* $\mathbf{x}(u,v)$.

**Proof.** Assume we have another parametrization $\bar{\mathbf{x}}(\bar{u},\bar{v}) : \bar{U} \subset \mathbb{R}^2 \to S$ such that $\bar{\mathbf{x}}(\bar{Q}) = \mathbf{x}(Q) = R$, where $\bar{Q} \subset \bar{U}$ is a compact region with boundary a piecewise smooth simple closed curve. Then we have $Q = \mathbf{x}^{-1} \circ \bar{\mathbf{x}}(\bar{Q})$ and due to the change of parameter function $h = \mathbf{x}^{-1} \circ \bar{\mathbf{x}}$ one can express $(u,v) \in Q = h(\bar{Q})$ as a function of $(\bar{u},\bar{v}) \in \bar{Q}$ (such a relation is a **diffeomorphism**). We now have the following identities

$$\begin{cases} (u,v) = h(\bar{u},\bar{v}) = (u(\bar{u},\bar{v}), v(\bar{u},\bar{v})), \\ \bar{\mathbf{x}}(\bar{u},\bar{v}) = \mathbf{x} \circ h(\bar{u},\bar{v}) = \mathbf{x}(u(\bar{u},\bar{v}), v(\bar{u},\bar{v})), \end{cases}$$

and by the chain rule

$$\bar{\mathbf{x}}_{\bar{u}} = \frac{\partial u}{\partial \bar{u}}\mathbf{x}_u + \frac{\partial v}{\partial \bar{u}}\mathbf{x}_v, \quad \bar{\mathbf{x}}_{\bar{v}} = \frac{\partial u}{\partial \bar{v}}\mathbf{x}_u + \frac{\partial v}{\partial \bar{v}}\mathbf{x}_v,$$

we get

$$\begin{aligned} \bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}} &= \left(\frac{\partial u}{\partial \bar{u}}\mathbf{x}_u + \frac{\partial v}{\partial \bar{u}}\mathbf{x}_v\right) \wedge \left(\frac{\partial u}{\partial \bar{v}}\mathbf{x}_u + \frac{\partial v}{\partial \bar{v}}\mathbf{x}_v\right) \\ &= \det\begin{pmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{pmatrix} \cdot (\mathbf{x}_u \wedge \mathbf{x}_v) \end{aligned} \tag{7}$$

More precisely, (7) is the same as

$$\begin{aligned} &\underbrace{\bar{\mathbf{x}}_{\bar{u}}(\bar{u},\bar{v}) \wedge \bar{\mathbf{x}}_{\bar{v}}(\bar{u},\bar{v})} \\ &= \det\begin{pmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{pmatrix} \cdot \left(\mathbf{x}_u(u(\bar{u},\bar{v}), v(\bar{u},\bar{v})) \wedge \mathbf{x}_v(u(\bar{u},\bar{v}), v(\bar{u},\bar{v}))\right) \\ &= \underbrace{\left(\mathbf{x}_u(h(\bar{u},\bar{v})) \wedge \mathbf{x}_v(h(\bar{u},\bar{v}))\right) \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}}, \quad \forall\ (\bar{u},\bar{v}) \in \bar{Q}. \end{aligned} \tag{8}$$

Now by the change of variables formula for multiple integrals (see the book by Marsden "**Elementary Classical Analysis, 2nd edition**", p. 523), we have (note that $Q = h\left(\bar{Q}\right)$)

$$\iint_Q |\mathbf{x}_u(u,v) \wedge \mathbf{x}_v(u,v)|\, dudv$$
$$= \iint_{\bar{Q}} |\mathbf{x}_u(h(\bar{u},\bar{v})) \wedge \mathbf{x}_v(h(\bar{u},\bar{v}))| \left|\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right| d\bar{u}d\bar{v} = \iint_{\bar{Q}} |\bar{\mathbf{x}}_{\bar{u}}(\bar{u},\bar{v}) \wedge \bar{\mathbf{x}}_{\bar{v}}(\bar{u},\bar{v})|\, d\bar{u}d\bar{v} \tag{9}$$

and we conclude

$$\iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v|\, dudv = \iint_{\bar{Q}} |\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}|\, d\bar{u}d\bar{v}, \tag{10}$$

i.e., the above definition is independent of the parametrizations we used. □

**Remark 1.16** *The assumption that the domain $R \subset S$ is contained in the image of a single parametrization is not very serious since in most examples there exists a parametrization $\mathbf{x}$ which cover the entire surface except for some curves, which do not contribute to the area.*

**Remark 1.17** *By the identity*

$$|\mathbf{x}_u \wedge \mathbf{x}_v|^2 + \langle \mathbf{x}_u, \mathbf{x}_v\rangle^2 = |\mathbf{x}_u|^2 |\mathbf{x}_v|^2,$$

*we have*

$$|\mathbf{x}_u \wedge \mathbf{x}_v|^2 = |\mathbf{x}_u|^2 |\mathbf{x}_v|^2 - \langle \mathbf{x}_u, \mathbf{x}_v\rangle^2,$$

*i.e.*

$$|\mathbf{x}_u \wedge \mathbf{x}_v| = \sqrt{EG - F^2}.$$

*Hence we can express $A(R)$ as*

$$A(R) = \iint_Q \sqrt{EG-F^2} dudv = A(R) = \iint_Q \sqrt{\begin{vmatrix} \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{vmatrix}} dudv. \tag{11}$$

**Example 1.18** *Do Example 5 in p. 101.*

## 1.2 Gradient on Surfaces (this is Exercise 14 in p. 104, Section 2-5, of the textbook).

**Definition 1.19** *The **gradient** of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map*

$$grad\ f : S \to \mathbb{R}^3, \tag{12}$$

*which assigns to each $p \in S$ a vector grad $f(p) \in T_pS \subset \mathbb{R}^3$ such that (the following identity is the **definition** of grad $f(p)$)*

$$\langle grad\ f(p), v\rangle_p = df_p(v) \quad \text{for all} \quad v \in T_pS. \tag{13}$$

*By property in linear algebra, the vector grad $f(p)$ satisfying (13) **exists** and is **unique**. If there are two vectors $v_1,\ v_2 \in T_pS$ which satisfies*

$$\langle v_1, v\rangle = df_p(v) \quad \text{for all} \quad v \in T_pS$$

*and*

$$\langle v_2, v\rangle = df_p(v) \quad \text{for all} \quad v \in T_pS,$$

*then we must have $v_1 = v_2$. Geometrically, one can view grad $f : S \to \mathbb{R}^3$ as a **vector field** on $S$. It assigns each $p \in S$ a vector grad $f(p) \in T_pS$.*

**Remark 1.20** *For simplicity, we usually use the notation* $\nabla_S f : S \to T_pS \subset \mathbb{R}^3$ *to denote the gradient of* $f : S \to \mathbb{R}$ *on the surface* $S$. *Therefore, we have*

$$\langle \nabla_S f(p), v \rangle = df_p(v) \quad \text{for all} \quad v \in T_pS. \tag{14}$$

Since $\nabla_S f$ is a **vector field** on $S$, it can be expressed as a linear combination of the basis $\{\mathbf{x}_u, \ \mathbf{x}_v\}$. Write

$$\nabla_S f = a\mathbf{x}_u + b\mathbf{x}_v.$$

Then

$$f_u = df_p(\mathbf{x}_u) = \langle \nabla_S f, \ \mathbf{x}_u \rangle = \langle a\mathbf{x}_u + b\mathbf{x}_v, \ \mathbf{x}_u \rangle = aE + bF, \quad f_u = \frac{\partial}{\partial u} f(\mathbf{x}(u,v)). \tag{15}$$

Similarly we have

$$f_v = df_p(\mathbf{x}_v) = \langle \nabla_S f, \ \mathbf{x}_v \rangle = \langle a\mathbf{x}_u + b\mathbf{x}_v, \ \mathbf{x}_v \rangle = aF + bG, \quad f_u = \frac{\partial}{\partial u} f(\mathbf{x}(u,v)). \tag{16}$$

Hence we get the matrix relation

$$\begin{pmatrix} f_u \\ f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} f_u \\ f_v \end{pmatrix}$$

and obtain, under a local parametrization $\mathbf{x}(u,v)$, the expression

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} f_u \\ f_v \end{pmatrix},$$

which implies

$$\nabla_S f = a\mathbf{x}_u + b\mathbf{x}_v = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{x}_v. \tag{17}$$

In particular, we see that $\nabla_S f$ is a differentiable function on $(u,v) \in U$. If $S = \mathbb{R}^2$ with Euclidean coordinates $x, \ y$, then we have $E = 1, \ F = 0, \ G = 1$, and the above becomes

$$\nabla_S f = \left(\frac{\partial f}{\partial x}\right)\mathbf{e}_1 + \left(\frac{\partial f}{\partial y}\right)\mathbf{e}_2$$

where $\{\mathbf{e}_1, \ \mathbf{e}_2\}$ is the standard basis of $\mathbb{R}^2$. Moreover, for a given regular surface $S$ **with orthogonal parametrization** $\mathbf{x}(u,v) : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$, (17) becomes (now we have $F \equiv 0$)

$$\nabla_S f = \frac{f_u}{E}\mathbf{x}_u + \frac{f_v}{G}\mathbf{x}_v. \tag{18}$$

Fix $p \in S$ and vary $v$ in the **unit circle** $|v| = 1$ in $T_pS$ centered at $p \in S$ (denote this compact set as $S^1 \subset T_pS$), then $df_p(v) = \langle \nabla_S f(p), v \rangle \in \mathbb{R}$ attains its maximum value over $S^1 \subset T_pS$ at

$$v = \frac{\nabla_S f(p)}{|\nabla_S f(p)|}, \quad df_p(v) = |\nabla_S f(p)| \tag{19}$$

and attains its minimum value over $S^1 \subset T_pS$ at

$$v = -\frac{\nabla_S f(p)}{|\nabla_S f(p)|}, \quad df_p(v) = -|\nabla_S f(p)|. \tag{20}$$

This is clear from (13).

### 1.2.1 Comparing Euclidean Gradient and Surface Gradient.

Assume $f:\mathbb{R}^3\to\mathbb{R}$ is a differentiable function and $S\subset\mathbb{R}^3$ is a regular surface. We can restrict $f$ onto $S$ and $f|_S:S\to\mathbb{R}$ is also a differentiable function on $S$. For fixed $p\in S$ we have two gradient vectors at $p$, namely $\nabla f(p)$ (a vector in the space $T_p\mathbb{R}^3\approx\mathbb{R}^3$ or one can say it is a vector in the ambient space $\mathbb{R}^3$) and $\nabla_S f(p)$ (a vector in the tangent space $T_pS\approx\mathbb{R}^2$). We have the following important result:

**Theorem 1.21** *Let $f:\mathbb{R}^3\to\mathbb{R}$ be a differentiable function and $S\subset\mathbb{R}^3$ be a regular surface. The* ***projection*** *of the vector $\nabla f(p)\in\mathbb{R}^3$ onto $T_pS$ is equal to $\nabla_S f(p)\in T_pS$.*

**Proof.** By the definition of gradient vector, we have the following:

$$\begin{cases} (1).\ \langle\nabla f(p),\ v\rangle = df_p(v), & \forall\, v\in\mathbb{R}^3, \\ (2).\ \langle\nabla_S f(p),\ v\rangle = df_p(v), & \forall\, v\in T_pS. \end{cases} \tag{21}$$

If we restrict $v$ onto the subspace $T_pS\subset\mathbb{R}^3$, then (1) implies

$$\langle\nabla f(p),\ v\rangle = df_p(v),\quad \forall\, v\in T_pS. \tag{22}$$

Since $v\in T_pS$ in (22), we have

$$\begin{aligned} &\langle\nabla f(p),\ v\rangle \\ &= \langle(\text{normal part of }\nabla f(p) + \text{tangential part of }\nabla f(p)),\ v\rangle \\ &= \langle\text{tangential part of }\nabla f(p),\ v\rangle,\quad \forall\, v\in T_pS. \end{aligned} \tag{23}$$

Therefore, we conclude

$$df_p(v) = \langle\nabla f(p),\ v\rangle = \left\langle \underbrace{\text{tangential part of }\nabla f(p)},\ v\right\rangle,\quad \forall\, v\in T_pS. \tag{24}$$

By (24) and (2) in (21) and uniqueness of gradient vector, we have

$$\nabla_S f(p) = \underbrace{\text{tangential part of }\nabla f(p)} = \text{the projection of }\nabla f(p)\ \text{ onto } T_pS. \tag{25}$$

The proof is done. □

## 1.3 Stereographic Projection of $S^2$ (this is Exercise 16 in p. 69, Section 2-2, of the textbook).

Consider the sphere $S^2$ given by

$$x^2+y^2+(z-1)^2=1.$$

Its north pole $N$ has coordinate $(0,0,2)$. For any $(x,y,z)\in S^2$, consider the line $L$ joining $N$ and $(x,y,z)$. This line $L$ will intersect the $xy$-plane at a unique point $(u,v)\in\mathbb{R}^2$. The map $\pi:(x,y,z)\in S^2\setminus(0,0,2)\to(u,v)\in\mathbb{R}^2$ is called **stereographic projection** of $S^2$. To describe it, it is easier to look at its inverse $\pi^{-1}$. Using comparison between two right triangles, we get (here $(x,y,z)\in S^2$)

$$\frac{2-z}{2} = \frac{\sqrt{x^2+y^2}}{\sqrt{u^2+v^2}} = \frac{\sqrt{1-(z-1)^2}}{\sqrt{u^2+v^2}}$$

and let $2-z=\lambda$ ($z=2-\lambda$) to get

$$\frac{\lambda}{2} = \frac{\sqrt{2\lambda-\lambda^2}}{\sqrt{u^2+v^2}}.$$

This gives $(u^2+v^2+4)\lambda^2 = 8\lambda$ and then

$$\lambda = \frac{8}{u^2+v^2+4}, \qquad z = 2-\lambda = 2 - \frac{8}{u^2+v^2+4} = \frac{2(u^2+v^2)}{u^2+v^2+4}$$

For $y$, we project these two right triangles onto $yz$-plane to get

$$\frac{2-z}{2} = \frac{\sqrt{y^2}}{\sqrt{v^2}} = \frac{y}{v} \quad (\text{if } y>0),$$

which gives

$$2 - \frac{2(u^2+v^2)}{u^2+v^2+4} = \frac{2y}{v}, \qquad y = \frac{4v}{u^2+v^2+4}.$$

Similarly, we project these two right triangles onto $xz$-plane to get

$$\frac{2-z}{2} = \frac{\sqrt{x^2}}{\sqrt{u^2}} = \frac{x}{u} \quad (\text{if } x>0),$$

which gives

$$x = \frac{4u}{u^2+v^2+4} := \frac{4u}{H}, \quad \text{where } H = u^2+v^2+4.$$

We conclude

$$\begin{aligned}\mathbf{x}(u,v) &:= \pi^{-1}(u,v) = (x(u,v), y(u,v), z(u,v)) \\ &= \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right) \\ &= \frac{1}{H}\left(4u,\ 4v,\ 2\left(u^2+v^2\right)\right), \quad (u,v)\in\mathbb{R}^2.\end{aligned}$$

One can check that $\mathbf{x}$ is a **homeomorphism** from $\mathbb{R}^2$ onto $S^2-\{N\}$, **differentiable** from $\mathbb{R}^2$ into $\mathbb{R}^3$. Moreover

$$\begin{aligned}\frac{\partial\mathbf{x}}{\partial u}(u,v) &= \frac{1}{H^2}\left(-4u^2+4v^2+16,\ -8uv,\ 16u\right), \quad H = u^2+v^2+4 \\ \frac{\partial\mathbf{x}}{\partial v}(u,v) &= \frac{1}{H^2}\left(-8uv,\ 4u^2-4v^2+16,\ 16v\right)\end{aligned}$$

and so

$$\begin{aligned}&\frac{\partial\mathbf{x}}{\partial u}(u,v)\wedge\frac{\partial\mathbf{x}}{\partial v}(u,v) \\ &= \frac{1}{H^4}\left(-64u\left(u^2+v^2+4\right),\ -64v\left(u^2+v^2+4\right),\ 16\left[16-\left(u^2+v^2\right)^2\right]\right).\end{aligned}$$

In particular, we also note that

$$\begin{aligned}&\left\langle\frac{\partial\mathbf{x}}{\partial u}(u,v), \frac{\partial\mathbf{x}}{\partial v}(u,v)\right\rangle \\ &= \frac{1}{H^4}\left[\left(-4u^2+4v^2+16\right)(-8uv)+(-8uv)\left(4u^2-4v^2+16\right)+(16u)(16v)\right] \\ &= 0, \quad \forall\ (u,v)\in\mathbb{R}^2.\end{aligned}$$

We easily see that $\frac{\partial\mathbf{x}}{\partial u}(u,v)\wedge\frac{\partial\mathbf{x}}{\partial v}(u,v)\neq(0,0,0)$ for all $(u,v)\in\mathbb{R}^2$ (because $\frac{\partial\mathbf{x}}{\partial u}(u,v)\perp\frac{\partial\mathbf{x}}{\partial v}(u,v)$ and both are nonzero vectors). Finally we see that one can choose two stereographic projections to cover the whole $S^2$. One misses the north pole and the other misses the south pole.

Finally, we compute the coefficients $E,\ F,\ G$ of the first fundamental form. We have

$$\begin{aligned} E(u,v) &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle \\ &= \frac{1}{H^4}\left[\left(-4u^2+4v^2+16\right)^2 + (-8uv)^2 + (16u)^2\right] = \frac{16}{H^2} \end{aligned} \tag{26}$$

and

$$\begin{aligned} G(u,v) &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\ &= \frac{1}{H^4}\left[(-8uv)^2 + \left(4u^2-4v^2+16\right)^2 + (16v)^2\right] = \frac{16}{H^2} \end{aligned} \tag{27}$$

and

$$F(u,v) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0. \tag{28}$$

We shall see later on that this parametrization $\mathbf{x}(u,v) : \mathbb{R}^2 \to S^2 \backslash \{(0,0,2)\}$ is a **conformal diffeomorphism** due to

$$E(u,v) = G(u,v) = \frac{16}{H^2}, \quad F(u,v) = 0, \quad \forall\ (u,v) \in \mathbb{R}^2 \text{ (domain of } \mathbf{x}\text{)}. \tag{29}$$

# 2 Chapter 3: The Geometry of the Gauss Map.

## 2.1 The Definition of the Gauss Map and its Fundamental Properties (this is Section 3-2 of the book).

### 2.1.1 Orientation of a Regular Surface $S \subset \mathbb{R}^3$.

We first recall the following fact: Let $\mathbf{x} : U$ (open set) $\subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a parametrization of a regular surface. The vector

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q) \in \mathbb{R}^3, \quad q \in U \tag{30}$$

is called **the unit normal vector field** on $\mathbf{x}(U) \subset S$ **induced by the parametrization** $\mathbf{x}$. Note: $N(q)$ is normal to $S$ at the point $p = \mathbf{x}(q)$ and $N : \mathbf{x}(U) \subset S \to \mathbb{R}^3$ is a **differentiable map** on $\mathbf{x}(U)$.

**Remark 2.1** *Since I did not teach Section 2.6 of the book, I will adopt the following definition for a regular surface $S \subset \mathbb{R}^3$ to be* ***orientable.***

**Definition 2.2** *A regular surface $S \subset \mathbb{R}^3$ is said to be* ***orientable*** *if there exists a* ***differentiable field of unit normal vectors*** *$N : S \to \mathbb{R}^3$ on the* ***whole*** *surface $S$. For simplicity, we just call it a* ***unit normal vector field on*** *$S$.*

**Remark 2.3** *There exist regular surfaces in $\mathbb{R}^3$ which are not orientable. One famous example is the* ***Möbius strip****. See the discussion in p. 108 for it (we will not discuss it).*

**Remark 2.4** *If there exists a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ such that $\mathbf{x}(U) = S$, then* ***the unit normal vector field*** *$N$ given by (30) is defined on the* ***whole*** *surface $S$. In such a case, $S$ is orientable. In particular, if $S$ is the graph of a differentiable function $f(x,y)$ defined on some open set $U$ of $\mathbb{R}^2$, then it is orientable.*

**Remark 2.5** *We know that every regular surface $S$ is* ***locally orientable****. Hence whether $S$ is orientable or not is a* ***global*** *property.*

In case $S$ is orientable, a choice of $N$ on $S$ is called an **orientation** on $S$ (this is equivalent to a choice of **compatible** coordinate neighborhoods covering $S$). If $S$ is connected, there are only two choices of $N$ on it. Therefore, if $S$ is connected, then it has exactly **two** orientations. If $S$ has $k$ connected components, then it has $2^k$ orientations.

If $S$ is a regular surface with an orientation $N$, a basis $\{v, w\}$ on $T_pS$ is called **positive** if $v \wedge w$ is pointing in the direction of $N(p)$, i.e. $\det(v, w, N(p)) > 0$. Otherwise, we say it is **negative**. If $\{v, w\}$ and $\{\tilde{v}, \tilde{w}\}$ are two positive bases on $T_pS$, then its change of coordinates has **positive determinant**.

The following says that the inverse image of a regular value of a differentiable function is also orientable. Thus, in general, it is difficult to find nonorientable surfaces in $\mathbb{R}^3$.

**Lemma 2.6** *Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and $a \in f(U)$ is a **regular value** of $f$. Then the surface*

$$S = \{(x, y, z) \in U : f(x, y, z) = a\}$$

*is orientable.*

**Proof.** We know that the gradient vector (it is a **nonzero** vector since $a$ is a regular value)

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}(x, y, z), \ \frac{\partial f}{\partial y}(x, y, z), \ \frac{\partial f}{\partial y}(x, y, z)\right), \quad (x, y, z) \in S$$

is everywhere perpendicular to $S$. Thus

$$N(x, y, z) = \frac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|}, \quad (x, y, z) \in S$$

is a unit normal field on $S$. By Definition 2.2, $S$ is orientable. □

Another important result is the following:

**Lemma 2.7** *Any **compact** regular surface $S \subset \mathbb{R}^3$ is orientable. Therefore, spheres and ellipsoids in $\mathbb{R}^3$ are both orientable.*

**Proof.** Omit it. □

### 2.1.2 Gauss Map of a Regular Surface $S \subset \mathbb{R}^3$.

**Throughout this chapter (Chapter 3 of the textbook), unless otherwise stated, we always assume that $S$ is orientable with a chosen orientation $N$. For simplicity, we call $S$ a regular surface with an orientation $N$.**

**Definition 2.8** *Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation $N$. The map $N : S \to \mathbb{R}^3$ has its values in the unit sphere*

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}.$$

*Thus we can write it as $N : S \to S^2$ and call it the **Gauss map** of $S$.*

**Remark 2.9** *Since $N : S \to \mathbb{R}^3$ is differentiable (this is due to definition), by p. 77, Example 3, we know that $N : S \to S^2$ is also differentiable. One can also use local parametrization to verify this. Its differential $dN_p : T_pS \to T_{N(p)}S^2$ can be viewed as a linear map from $T_pS \to T_pS$ because one can identify $T_{N(p)}S^2$ and $T_pS$ (they are parallel planes in $\mathbb{R}^3$; both planes are normal to $N(p)$). For any $v \in T_pS$, choose $\alpha(t) \in S$, $t \in (-\varepsilon, \varepsilon)$, so that $\alpha(0) = p$, $\alpha'(0) = v$. Then we have*

$$dN_p(v) = \left.\frac{d}{dt}\right|_{t=0} N(\alpha(t)), \tag{31}$$

*or more generally,*

$$dN_{\alpha(t)}\left(\alpha'(t)\right)=\frac{d}{dt}N\left(\alpha(t)\right),\quad \forall\, t\in(-\varepsilon,\varepsilon).$$

*If we write $\alpha(t)$ as $\mathbf{x}(u(t),v(t))$ with $\alpha(0)=\mathbf{x}(u(0),v(0))=p$ then*

$$N\left(\alpha(t)\right)=N\left(\mathbf{x}\left(u(t),v(t)\right)\right)$$

*and for simplicity we will just write $N(\mathbf{x}(u(t),v(t)))$ as $N(u(t),v(t))$ with the understanding that $N(u,v)$ is actually $N(\mathbf{x}(u,v))$. By this, we have*

$$\frac{d}{dt}N\left(\alpha(t)\right)=\frac{d}{dt}N\left(u(t),v(t)\right)=N_u\left(u(t),v(t)\right)u'(t)+N_v\left(u(t),v(t)\right)v'(t),\quad \forall\, t\in(-\varepsilon,\varepsilon)$$

*and*

$$\left.\frac{d}{dt}\right|_{t=0}N\left(\alpha(t)\right)=\underbrace{N_u(p)\,u'(0)+N_v(p)\,v'(0)}.$$

*On the other hand, we can also write the above $\left.\frac{d}{dt}\right|_{t=0}N(\alpha(t))$ as*

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=0}N\left(\alpha(t)\right)\\&=dN_p\left(\alpha'(0)\right)=dN_p\left[u'(0)\,\mathbf{x}_u+v'(0)\,\mathbf{x}_v\right]=\underbrace{u'(0)\,dN_p\left(\mathbf{x}_u\right)+v'(0)\,dN_p\left(\mathbf{x}_v\right)}.\end{aligned}$$

*In particular, we note that*

$$N_u(p)=dN_p\left(\mathbf{x}_u\right),\qquad N_v(p)=dN_p\left(\mathbf{x}_v\right), \tag{32}$$

*where $\mathbf{x}_u$ and $\mathbf{x}_v$ in (32) are evaluated at $(u(0),v(0))\in U$.*

**Example 2.10** *Do Example 2 in p. 139.*

**Example 2.11** *Do Example 3 in p. 141. In this example, we have*

$$dN\left(x'(t),y'(t),z'(t)\right)=\left(-x'(t),-y'(t),0\right),$$

*which can be written as*

$$dN\left[\left(x'(t),y'(t),0\right)+\left(0,0,z'(t)\right)\right]=\left(-x'(t),-y'(t),0\right).$$

*Thus the differential $dN_p:T_pS\to T_pS$ has two eigenvalues $0$ and $-1$.*

**Example 2.12** *Do Example 4 in p. 141. In this example, we have $\mathbf{x}(u,v)=(u,v,v^2-u^2)$, $(u,v)\in\mathbb{R}^2$, and we choose*

$$\begin{aligned}N(u,v)=N\left(\mathbf{x}(u,v)\right)&=\frac{\mathbf{x}_u\wedge\mathbf{x}_v}{\left|\mathbf{x}_u\wedge\mathbf{x}_v\right|}(u,v)\\&=\left(\frac{u}{\sqrt{u^2+v^2+\frac14}},\frac{-v}{\sqrt{u^2+v^2+\frac14}},\frac{1}{2\sqrt{u^2+v^2+\frac14}}\right).\end{aligned}$$

*Hence at $p=(0,0,0)=\mathbf{x}(0,0)$ we have $\mathbf{x}_u(0,0)=(1,0,0)$ and $\mathbf{x}_v(0,0)=(0,1,0)$ and*

$$\begin{aligned}dN_p\left((1,0,0)\right)=dN_p\left(\mathbf{x}_u(0,0)\right)&=\left.\frac{\partial}{\partial u}\right|_{u=0}N\left(\mathbf{x}(u,0)\right)=\left.\frac{\partial}{\partial u}\right|_{u=0}N(u,0)\\&=\left.\frac{\partial}{\partial u}\right|_{u=0}\left(\frac{u}{\sqrt{u^2+\frac14}},0,\frac{1}{2\sqrt{u^2+\frac14}}\right)=(2,0,0),\end{aligned}$$

*and similarly*

$$dN_p((0,1,0)) = dN_p(\mathbf{x}_v(0,0)) = \left.\frac{\partial}{\partial v}\right|_{v=0} N(\mathbf{x}(0,v)) = \left.\frac{\partial}{\partial v}\right|_{v=0} N(0,v)$$
$$= \left.\frac{\partial}{\partial v}\right|_{v=0} \left(0, \frac{-v}{\sqrt{v^2+\frac{1}{4}}}, \frac{1}{2\sqrt{v^2+\frac{1}{4}}}\right) = (0,-2,0).$$

*In general, we have*

$$\left.\frac{d}{dt}\right|_{t=0} N(u(t),v(t)) = \left.\frac{d}{dt}\right|_{t=0} N(\mathbf{x}(u(t),v(t)))$$
$$= dN_p(u'(0)\mathbf{x}_u(0,0) + v'(0)\mathbf{x}_v(0,0)) = (2u'(0), -2v'(0), 0),$$

*i.e.,*

$$dN_p(u'(0), v'(0), 0) = (2u'(0), -2v'(0), 0).$$

*Thus* $dN_p : T_pS \to T_pS$ *has two eigenvalues* $2$ *and* $-2$ *with corresponding eigenvectors* $\mathbf{x}_u(0,0) = (1,0,0)$ *and* $\mathbf{x}_v(0,0) = (0,1,0)$.

**Example 2.13** *Do Example 5 in p. 140. In this example, we have* $\mathbf{x}(u,v) = (u, v, u^2 + kv^2)$, $(u,v) \in \mathbb{R}^2$, *where*

$$\begin{cases} \mathbf{x}_u(u,v) = (1,0,2u), \quad \mathbf{x}_v(u,v) = (0,1,2kv), \\ \mathbf{x}_u(u,v) \wedge \mathbf{x}_v(u,v) = (-2u, -2kv, 1), \end{cases}$$

*and we choose*

$$N(u,v) = N(\mathbf{x}(u,v)) = -\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(u,v)$$
$$= \left(\frac{u}{\sqrt{u^2+k^2v^2+\frac{1}{4}}}, \frac{kv}{\sqrt{u^2+k^2v^2+\frac{1}{4}}}, \frac{-1}{2\sqrt{u^2+k^2v^2+\frac{1}{4}}}\right).$$

*Hence at* $p = (0,0,0)$ *we have* $\mathbf{x}_u(0,0) = (1,0,0)$ *and* $\mathbf{x}_v(0,0) = (0,1,0)$ *and*

$$dN_p((1,0,0)) = dN_p(\mathbf{x}_u(0,0)) = \left.\frac{\partial}{\partial u}\right|_{u=0} N(u,0) = (2,0,0),$$

*and similarly*

$$dN_p((0,1,0)) = dN_p(\mathbf{x}_v(0,0)) = \left.\frac{\partial}{\partial v}\right|_{v=0} N(0,v) = (0,2k,0).$$

*In general, we have*

$$\left.\frac{d}{dt}\right|_{t=0} N(u(t),v(t)) = \left.\frac{d}{dt}\right|_{t=0} N(\mathbf{x}(u(t),v(t)))$$
$$= dN_p[u'(0)\mathbf{x}_u(0,0) + v'(0)\mathbf{x}_v(0,0)] = (2u'(0), 2kv'(0), 0),$$

*i.e.,*

$$dN_p(u'(0), v'(0), 0) = (2u'(0), 2kv'(0), 0).$$

*Thus* $dN_p : T_pS \to T_pS$ *has two eigenvalues* $2$ *and* $2k$ *with corresponding eigenvectors* $\mathbf{x}_u(0,0) = (1,0,0)$ *and* $\mathbf{x}_v(0,0) = (0,1,0)$.

**Proposition 2.14** ***(This is Proposition 1 in p. 142.)*** *For each* $p \in S$. *the differential* $dN_p : T_pS \to T_pS$ *of the Gauss map is a* ***self-adjoint*** *linear map.*

**Remark 2.15** *Read the appendix "self-adjoint linear map and quadratic forms" in p.217-219 by yourself.*

**Remark 2.16** *Note that with respect to an* ***orthonormal basis*** $\{v_1, v_2\}$ *on* $T_pS$, *the matrix representation* $M$ *for* $dN_p : T_pS \to T_pS$ *is* ***symmetric****. If the basis* $\{v_1, v_2\}$ *is* ***not orthonormal****,* $M$ *may not be symmetric in general. In particular, the matrix for the linear map* $dN_p : T_pS \to T_pS$ *with respect to a parametrization basis* $\{\mathbf{x}_u, \mathbf{x}_v\}$ *may not be symmetric in general.*

**Proof.** Fix $p \in S$ and assume $\mathbf{x}(u,v) : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ is a parametrization of $S$ around $p$ with $\mathbf{x}(q) = p$. We already know that $dN_p : T_pS \to T_pS$ is linear and $\{\mathbf{x}_u, \mathbf{x}_v\}$ (evaluated at $q \in U$) is a basis on $T_pS$. To show that it is self-adjoint, we need to check that

$$\langle dN_p(v), w\rangle = \langle v, dN_p(w)\rangle \quad \text{for all} \quad v,\ w \in T_pS. \tag{33}$$

By linearity, it suffices to check that

$$\langle dN_p(\mathbf{x}_u), \mathbf{x}_v\rangle = \langle \mathbf{x}_u, dN_p(\mathbf{x}_v)\rangle, \tag{34}$$

which is the same as

$$\langle N_u(u,v), \mathbf{x}_v(u,v)\rangle = \langle N_v(u,v), \mathbf{x}_u(u,v)\rangle \quad \text{at} \quad q \in U, \tag{35}$$

where $N(u,v)$ means $N(\mathbf{x}(u,v))$, where $(u,v) \in U$.

Note that for a parametrization $\mathbf{x}(u,v)$ we have

$$\langle N(u,v), \mathbf{x}_u(u,v)\rangle = \langle N(u,v), \mathbf{x}_v(u,v)\rangle = 0 \quad \text{for all} \quad (u,v) \in U.$$

By differentiation with respect to $u$ and $v$ respectively, the above will imply

$$\begin{cases} \langle N_u(u,v), \mathbf{x}_u(u,v)\rangle = -\langle N(u,v), \mathbf{x}_{uu}(u,v)\rangle, & \text{where } N_u(u,v) = dN(\mathbf{x}_u(u,v)) \\ \langle N_v(u,v), \mathbf{x}_v(u,v)\rangle = -\langle N(u,v), \mathbf{x}_{vv}(u,v)\rangle, & \text{where } N_v(u,v) = dN(\mathbf{x}_v(u,v)) \\ \langle N_u(u,v), \mathbf{x}_v(u,v)\rangle = \langle N_v(u,v), \mathbf{x}_u(u,v)\rangle = -\langle N(u,v), \mathbf{x}_{uv}(u,v)\rangle, & \mathbf{x}_{uv}(u,v) = \mathbf{x}_{vu}(u,v) \end{cases}$$

for all $(u,v) \in U$. In particular, at the point $p \in S$, we have the identity (34).

The proof is done. □

Another useful result is the following:

**Lemma 2.17** *If the differential* $dN_p : T_pS \to T_pS$ *of the Gauss map satisfies*

$$\langle dN_p(v),\ v\rangle = 0, \quad \forall\ v \in T_pS, \tag{36}$$

*then* $dN_p(v) = 0$ *for all* $v \in T_pS$.

**Proof.** By the assumption, we have

$$\langle dN_p(v+w),\ v+w\rangle = 0, \quad \forall\ v,\ w \in T_pS,$$

which gives

$$\langle dN_p(v),\ w\rangle + \langle dN_p(w),\ v\rangle = 2\langle dN_p(v),\ w\rangle = 0, \quad \forall\ v,\ w \in T_pS.$$

Hence for fixed $v \in T_pS$ we have $\langle dN_p(v),\ w\rangle = 0$ for all $w \in T_pS$. This implies $dN_p(v) = 0$. But since $v \in T_pS$ can be arbitrary, we conclude $dN_p(v) = 0$ for all $v \in T_pS$. □

The most important property of a self-adjoint linear map from an **inner product vector space** $V$ with $\dim V = 2$ is the following:

**Theorem 2.18** *Let $V$ be an **inner product vector space** with $\dim V = 2$ and assume $A : V \to V$ is a **self-adjoint** linear map. Then there exists an orthonormal basis $\{e_1, e_2\}$ of $V$ such that*

$$Ae_1 = \lambda_1 e_1, \quad Ae_2 = \lambda_2 e_2 \tag{37}$$

*for some $\lambda_1,\ \lambda_2 \in \mathbb{R}$ (without loss of generality we may assume $\lambda_1 \geq \lambda_2$), i.e. $e_1,\ e_2$ are eigenvectors and $\lambda_1,\ \lambda_2$ are eigenvalues of $A$. Moreover, we have*

$$\lambda_1 = \max_{v \in V,\ |v|=1} \langle Av, v \rangle, \quad \lambda_2 = \min_{v \in V,\ |v|=1} \langle Av, v \rangle. \tag{38}$$

**Proof.** See p. 219 of the textbook. We omit it. □

**Remark 2.19** *(**Interesting observation.**) Assume $A : V \to V$ is a **self-adjoint** linear map with* $\dim V = 2$*. If $\lambda_1 > \lambda_2$ are two eigenvalues of $A$ with corresponding unit eigenvectors $v_1,\ v_2$, then we must have $v_1 \perp v_2$ and*

$$\lambda_1 = \max_{v \in V,\ |v|=1} \langle Av, v \rangle, \quad \lambda_2 = \min_{v \in V,\ |v|=1} \langle Av, v \rangle.$$

*To see this, by the identities*

$$(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = \langle Av_1, v_2 \rangle - \langle v_1, Av_2 \rangle = 0, \quad \lambda_1 - \lambda_2 \neq 0,$$

*we have $v_1 \perp v_2$. Finally, for any $v \in V$ with $|v| = 1$ we can express it as*

$$v = \alpha v_1 + \beta v_2 \quad \text{for some} \quad \alpha,\ \beta \in \mathbb{R}, \quad \alpha^2 + \beta^2 = 1$$

*and by*

$$\langle Av, v \rangle = \langle \alpha\lambda_1 v_1 + \beta\lambda_2 v_2,\ \alpha v_1 + \beta v_2 \rangle = \lambda_1\alpha^2 + \lambda_2\beta^2,$$

*where*

$$\lambda_2 = \lambda_2\alpha^2 + \lambda_2\beta^2 \leq \lambda_1\alpha^2 + \lambda_2\beta^2 \leq \lambda_1\alpha^2 + \lambda_1\beta^2 = \lambda_1,$$

*we obtain*

$$\lambda_1 = \max_{v \in V,\ |v|=1} \langle Av, v \rangle, \quad \lambda_2 = \min_{v \in V,\ |v|=1} \langle Av, v \rangle. \tag{39}$$

### 2.1.3 Second Fundamental Form, Normal Curvature, and Geodesic Curvature.

**Definition 2.20** *The quadratic form $II_p(v) := -\langle dN_p(v), v \rangle : T_pS \to \mathbb{R}$ is called the **second fundamental form** of $S$ at $p$. Note that we have*

$$II_p(-v) = II_p(v) \quad \text{for all} \quad v \in T_pS \tag{40}$$

*and the linear map $dN_p : T_pS \to T_pS$ is self-adjoint.*

**Definition 2.21** *Let $C$ be a **regular curve** in $S$ (with unit normal $N$) parametrized by $\alpha(s)$ (with Frenet frame $\{t(s),\ n(s),\ b(s)\}$), where $s \in I$ is arc length parameter. The **unit vector***

$$\mathbf{n}_{int}(s) = N(s) \wedge \alpha'(s) = N(s) \wedge t(s), \quad s \in I \quad (N(s) \text{ means } N(\alpha(s))) \tag{41}$$

*is called the **intrinsic normal** of $\alpha$ at $s$. It is an unit vector **lying on** $T_{\alpha(s)}S$ and **normal to the curve** $\alpha$ at $s$, i.e.*

$$\mathbf{n}_{int}(s) \in T_{\alpha(s)}S, \quad \langle \mathbf{n}_{int}(s), t(s) \rangle = \langle \mathbf{n}_{int}(s), \alpha'(s) \rangle = 0, \quad s \in I. \tag{42}$$

*That is why we call it the **intrinsic normal** of $\alpha$. Now the three vectors*

$$\{t(s),\ \mathbf{n}_{int}(s),\ N(s)\} \tag{43}$$

*form an **orthonormal frame** at $\alpha(s)$ (it also has the the previous **Frenet frame** $\{t(s),\ n(s),\ b(s)\}$ at $\alpha(s)$). The tangent plane $T_{\alpha(s)}S$ is spanned by the orthonormal basis $\{t(s),\ \mathbf{n}_{int}(s)\}$ and*

$$t(s) \wedge \mathbf{n}_{int}(s) = N(s). \tag{44}$$

**Remark 2.22** *(Important.) Note that* $\mathbf{n}_{int}(s) = N(s) \wedge t(s)$ *is always defined even if* $n(s)$ *of* $\alpha$ *is undefined (i.e., when* $\alpha''(s) = 0$*).*

**Remark 2.23** *(Important.) The* ***projection*** *of the normal vector* $n(s) \in \mathbb{R}^3$ *of* $\alpha$ *onto the* ***tangent plane*** $T_{\alpha(s)}S$ *is given by:*

$$\begin{aligned} &\text{projection of } n(s) \text{ onto } T_{\alpha(s)}S \\ &= \underbrace{\langle n(s), t(s)\rangle}\, t(s) + \underbrace{\langle n(s), \mathbf{n}_{int}(s)\rangle}\, \mathbf{n}_{int}(s) = \underbrace{\langle n(s), \mathbf{n}_{int}(s)\rangle}\, \mathbf{n}_{int}(s). \end{aligned} \tag{45}$$

*This explains why we call* $\mathbf{n}_{int}(s)$ *the* ***intrinsic normal*** *of* $\alpha$ *because it is the projection of* $n(s)$ *onto* $T_{\alpha(s)}S$*.*

**Remark 2.24** *(Important.) To study the geometry of a curve* $\alpha(s)$ *on* $S$*, it is better to use the frame* $\{t(s),\ \mathbf{n}_{int}(s),\ N(s)\}$ *(it respects the surface) than the Frenet frame* $\{t(s),\ n(s),\ b(s)\}$*.*

For a parametrized curve $\alpha(s) \in S$ (with normal $N$), $s \in I$, since we have $\langle \alpha''(s),\ \alpha'(s)\rangle = 0$ everywhere, the vector $\alpha''(s)$ must lie on the plane spanned by $\mathbf{n}_{int}(s)$ and $N(s)$. Therefore, we have

$$\alpha''(s) = k(s)\, n(s) = \underbrace{\langle \alpha''(s),\ \mathbf{n}_{int}(s)\rangle}\, \mathbf{n}_{int}(s) + \underbrace{\langle \alpha''(s),\ N(s)\rangle}\, N(s), \quad s \in I. \tag{46}$$

**Definition 2.25** *In (46), the quantity*

$$\langle \alpha''(s),\ \mathbf{n}_{int}(s)\rangle \ \ \text{(denoted as } k_g(s)\text{)}$$

*is called the* ***geodesic curvature*** *of* $\alpha$*; and the quantity*

$$\langle \alpha''(s),\ N(s)\rangle \ \ \text{(denoted as } k_n(s)\text{)}$$

*is called the* ***normal curvature*** *of* $\alpha$*. Therefore, we have the identity*

$$\alpha''(s) = k(s)\, n(s) = k_g(s)\, \mathbf{n}_{int}(s) + k_n(s)\, N(s), \quad s \in I.$$

*Since we also have* $|\alpha''(s)| = k(s) \geq 0$ *(the curvature of* $\alpha(s)$ *as a curve in* $\mathbb{R}^3$*), we conclude the important identity*

$$k^2(s) = |\alpha''(s)|^2 = k_g^2(s) + k_n^2(s), \quad s \in I, \tag{47}$$

*which is the same as*

$$k(s) = \sqrt{k_g^2(s) + k_n^2(s)}, \quad s \in I.$$

*If* $\alpha''(s) \neq 0$ *(then* $n(s)$ *is defined), we also have*

$$\begin{cases} k_g(s) = \underbrace{\langle \alpha''(s), \mathbf{n}_{int}(s)\rangle} = k(s)\langle n(s), \mathbf{n}_{int}(s)\rangle, & \text{where } \mathbf{n}_{int}(s) = N(s) \wedge \alpha'(s) \\ k_n(s) = \underbrace{\langle \alpha''(s), N(s)\rangle} = k(s)\langle n(s), N(s)\rangle, & s \in I. \end{cases} \tag{48}$$

*Finally, if* $\alpha''(s) = 0$ *(then* $n(s)$ *is undefined), we have* $k_g(s) = k_g(s) = k(s) = 0$*.*

**Remark 2.26** *If we change the orientation of* $S$*, then both* $\mathbf{n}_{int}(s)$ *and* $N(s)$ *change sign and so do* $k_n(s)$ *and* $k_g(s)$*.*

Assume $\alpha(0) = p \in S$. We have the following important identity:

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle = -\langle N'(0), \alpha'(0)\rangle = \langle N(0), \alpha''(0)\rangle \\ &= \langle N(0), k(0)\,\mathbf{n}(0)\rangle = k_n(0) \ \text{ (or denote it as } k_n(p)\text{ )} = \textbf{normal curvature} \text{ at } p. \end{aligned} \tag{49}$$

We conclude that the value of **the second fundamental form** $II_p(v)$ at a unit vector $v \in T_pS$ is equal to the **normal curvature** of a regular curve $\alpha$ passing through $p$ and tangent to $v$, i.e.

$$II_{\alpha(s)}(\alpha'(s)) = -\langle dN_p(\alpha'(s)), \alpha'(s)\rangle = k_n(s), \quad \forall\ s \in I. \tag{50}$$

**Thus the normal curvature of $\alpha$ at $p$ is actually to measure the "geometry" of $S$, NOT the curvature of $\alpha$.** It is the component of the tangent vector $-dN_p(\alpha'(s))$ in the direction $\alpha'(s)$, $s \in I$.

By the identity (49), we have the following important observation:

**Lemma 2.27** *The **normal curvature** $k_n(s)$ measures **the "geometry" of $S$ in $\mathbb{R}^3$** (along some direction $v \in T_pS$ given by $\alpha'(0)$) and the **geodesic curvature** $k_g(s)$ measures **how $\alpha$ is curving in $S$**. The geodesic curvature of $\alpha$ is the **intrinsic curvature** of $\alpha$ in $S$, i.e., the curvature viewed by the surface $S$. However, the **curvature** $k(s)$ of $\alpha \in S$ in $\mathbb{R}^3$ depends on **the "geometry" of $S$ in $\mathbb{R}^3$** and **how $\alpha$ is curving in $S$**, i.e. $k(s)$ depends on $k_n(s)$ and $k_g(s)$.*

Another interesting observation is the following:

**Proposition 2.28** ***(This is Proposition 2 in p. 144.) (Meusnier.)** All curves (parametrized by arc length parameter) lying on $S$ passing through $p \in S$ and having the same tangent line at $p$ have **the same normal curvature** at $p$.*

**Remark 2.29** ***(Important.)** By the above Proposition, we can talk about the **normal curvature along a given direction** $v \in T_pS$ at $p$ (**here both $v$ and $-v$ are regarded as having the same direction**). Moreover, if we change the direction $v$ into $-v$, we get the same normal curvature.*

**Proof.** Let $\alpha(s)$ and $\beta(s)$ be two regular curves lying on $S$ with $\alpha(0) = \beta(0) = p \in S$ and $\alpha'(0) = v \in T_pS$. By the assumption we have $\beta'(0) = \pm v$ and by (49), the normal curvature of $\alpha$ at $p$ is

$$k_n^{(\alpha)}(0) = II_p(\alpha'(0)) = II_p(v).$$

On the other hand, the normal curvature of $\beta$ at $p$ is

$$k_n^{(\beta)}(0) = II_p(\beta'(0)) = II_p(\pm v) = II_p(v).$$

The proof is done. □

**Example 2.30** ***(Curve lying on $S^2$.)** Let $S^2 \subset \mathbb{R}^3$ be the unit sphere in $\mathbb{R}^3$ centered at the origin $O = (0,0,0)$ with chosen orientation $N(p) = -p$ (inward) for all $p \in S^2$. Let $\alpha(s) : I \to S^2$ be a regular curve lying on $S^2$. We have $N(s) = -\alpha(s)$ and the following three useful identities*

$$\langle \alpha(s),\ \alpha(s)\rangle \equiv 1, \quad \langle \alpha(s),\ \alpha'(s)\rangle \equiv 0, \quad \langle \alpha'(s),\ \alpha'(s)\rangle \equiv 1, \quad \forall\ s \in I,$$

*which give*

$$\langle \alpha''(s),\ \alpha(s)\rangle = -\langle \alpha'(s),\ \alpha'(s)\rangle \equiv -1, \quad \langle \alpha''(s),\ \alpha'(s)\rangle \equiv 0, \quad \forall\ s \in I,$$

*By definition*

$$\mathbf{n}_{int}(s) = N(s) \wedge \alpha'(s) = -\alpha(s) \wedge \alpha'(s) \tag{51}$$

*and the frame* $\{t(s),\ \mathbf{n}_{int}(s),\ N(s)\}$ *is given by*

$$\{t(s),\ \mathbf{n}_{int}(s),\ N(s)\} = \{\alpha'(s),\ -\alpha(s)\wedge\alpha'(s),\ -\alpha(s)\}.$$

*The geodesic curvature and normal curvature are given by*

$$\begin{cases} k_g(s) = \langle\alpha''(s), \mathbf{n}_{int}(s)\rangle = -\langle\alpha''(s),\ \alpha(s)\wedge\alpha'(s)\rangle = -\det(\alpha(s), \alpha'(s), \alpha''(s)) \\ k_n(s) = \langle\alpha''(s), N(s)\rangle = \langle\alpha''(s),\ -\alpha(s)\rangle = \langle\alpha'(s),\ \alpha'(s)\rangle \equiv 1, \quad s\in I \end{cases} \tag{52}$$

*and the vector* $\alpha''(s)$ *can be decomposed as*

$$\begin{aligned} \alpha''(s) &= k(s)\,n(s) = k_g(s)\,\mathbf{n}_{int}(s) + k_n(s)\,N(s) \\ &= -\det(\alpha(s), \alpha'(s), \alpha''(s))\,\mathbf{n}_{int}(s) + 1\cdot N(s), \quad s\in I, \end{aligned} \tag{53}$$

*which gives the curvature identity*

$$k(s) = \sqrt{1+k_g^2(s)} = \sqrt{1+[\det(\alpha(s), \alpha'(s), \alpha''(s))]^2}, \quad s\in I. \tag{54}$$

*One can also compute* $k_n(s)$ *by the identity (49) and get*

$$\begin{aligned} k_n(s) &= -\left\langle dN_{\alpha(s)}(\alpha'(s)), \alpha'(s)\right\rangle \\ &= -\langle N'(s), \alpha'(s)\rangle = \langle\alpha'(s), \alpha'(s)\rangle \equiv 1, \quad \forall\ s\in I. \end{aligned}$$

*Finally, we note that if we change the orientation of* $S^2$ *by choosing* $N(p) = p$ *(outward) for all* $p\in S^2$*, both* $\mathbf{n}_{int}(s)$ *and* $N(s)$ *change sign and so do* $k_n(s)$ *and* $k_g(s)$*. In such a case, we have the nice identity*

$$k_g(s) = \det(\alpha(s), \alpha'(s), \alpha''(s)), \quad s\in I. \tag{55}$$

*This example confirms our observation in Lemma 2.27. The normal curvature* $k_n(s)\equiv 1$ *describes the geometry of* $S^2$*, not the geometry of* $\alpha(s)$*.*

**Example 2.31** *(**Curve lying on** $\mathbb{R}^2$.) If* $\alpha(s)$ *is a curve lying on the plane* $\mathbb{R}^2\subset\mathbb{R}^3$*, then* $k_n(s)\equiv 0$ *(since the plane has **no** "curvature") and* $k_g(s) = k(s)$*, where* $k(s)$ *is the **signed curvature** of* $\alpha$ *in the plane. More precisely, we choose* $N(p) = (0,0,1)$ *(upward) for all* $p\in\mathbb{R}^2$ *and so* $dN_p(v) = 0$ *for all* $p\in\mathbb{R}^2$ *and all* $v\in T_pS$*. If we write the unit tangent vector* $\alpha'(s)$ *as* $\alpha'(s) = (\cos\theta(s), \sin\theta(s), 0)$*, we get*

$$\mathbf{n}_{int}(s) = N(s)\wedge\alpha'(s) = (0,0,1)\wedge(\cos\theta(s), \sin\theta(s), 0) = (-\sin\theta(s), \cos\theta(s), 0)$$

*and then* $\mathbf{n}_{int}(s)$ *is the same as the normal vector* $n(s)$ *for plane curve* $\alpha(s)$*. Hence*

$$k_g(s) = \langle\alpha''(s), \mathbf{n}_{int}(s)\rangle = \langle\alpha''(s), n(s)\rangle = k(s) = \textbf{\textit{signed curvature}}\ \textit{of}\ \alpha(s). \tag{56}$$

There is other way to express the geodesic curvature $k_g(s)$. Assume the Frenet frame $\{t(s),\ n(s),\ b(s)\}$ of $\alpha(s)$ exists. We have

$$\begin{aligned} k_g(s) &= \langle\alpha''(s), \mathbf{n}_{int}(s)\rangle = k(s)\langle n(s), N(s)\wedge t(s)\rangle \\ &= k\langle N(s),\ t(s)\wedge n(s)\rangle = k(s)\langle b(s), N(s)\rangle \end{aligned} \tag{57}$$

and conclude

$$\begin{cases} k_n(s) = \langle\alpha''(s), N(s)\rangle = \underbrace{k(s)\langle n(s), N(s)\rangle} = \text{normal curvature}, \\ k_g(s) = \langle\alpha''(s), \mathbf{n}_{int}(s)\rangle = \underbrace{k(s)\langle b(s), N(s)\rangle} = \text{geodesic curvature}. \end{cases} \tag{58}$$

**Definition 2.32** *Let $S \subset \mathbb{R}^3$ be a regular surface with a chosen orientation $N$. Given a unit vector $v \in T_pS$, the set*

$$S \bigcap P$$

*is called the* ***normal section*** *of $S$ at $p$* ***in the direction*** *$v$. Here $P$ is the plane passing through $p$ and contains the two vectors $N(p)$ and $v \in T_pS$.*

Since $S$ is a regular surface and near $p$ it is like the graph of a function $z = f(x,y)$, $(x,y) \in U$ (some open set containing $(0,0)$) with $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$; see Exercise 26 in p. 93), therefore near $p$ the **normal section** of $S$ at $p$ (along any direction $v \in T_pS$) is a **regular plane curve** $C$ lying on $S$ (here "regular curve" is in the sense of p. 77-78 of the textbook). If we parametrize it by $\alpha(s)$, $s \in (-\varepsilon, \varepsilon)$, with $\alpha(0) = p$, then $\alpha(s)$ is a curve on $S$ (and also on the plane $P$) with $\alpha'(0) = v \in T_pS$ and either (in the following, the curvature $k(s)$ of $\alpha(s)$ is defined as $k(s) = |\alpha''(s)| \geq 0$)

$$\alpha''(0) \neq 0, \quad \alpha''(0) = k(0)\,\mathbf{n}(0) \quad \text{with} \quad k(0) = |\alpha''(0)| > 0 \tag{59}$$

or

$$\alpha''(0) = 0, \quad \text{with} \quad k(0) = |\alpha''(0)| = 0 \quad \text{and} \quad \mathbf{n}(0) \text{ is undefined.}$$

where in (59) the curvature $k(0)$ is defined as $|\alpha''(0)|$. In the first case, $\mathbf{n}(0)$ is defined with

$$\mathbf{n}(0) = N(0) \quad \text{or} \quad \mathbf{n}(0) = -N(0).$$

In the second case, $\mathbf{n}(0)$ is undefined.

By the relation $k_n(0) = \langle \alpha''(0), N(0) \rangle$ we have the following possibilities:

$$\begin{cases} k_n(0) = k(0) > 0 & \text{if} \quad \mathbf{n}(0) = N(0), \\ k_n(0) = -k(0) < 0 & \text{if} \quad \mathbf{n}(0) = -N(0), \\ k_n(0) = 0 & \text{if} \quad \alpha''(0) = 0 \ \ (\mathbf{n}(0) \text{ is undefined}) \end{cases} \tag{60}$$

In any case, we conclude (note that here, similar to space curves, $k(0)$ is defined as $|\alpha''(0)| \geq 0$)

$$|k_n(0)| = k(0). \tag{61}$$

In particular, we see that the **normal section** of $S$ at $p$ has **no geodesic curvature**.

**Remark 2.33** *The purpose of the normal section is to find a curve $\alpha(s)$, $\alpha(0) = p$, on $S$ such that its curvature $k(0)$ is equal to the absolute value of the* ***normal curvature*** *$k_n(0)$.*

**Example 2.34** *Let $S = S^2$ centered at $(0,0,0)$ with inward normal $N$ and $p \in S^2$. Each of a normal section of $S$ at $p$ in the direction $v \in T_pS$ is a great circle with $0$ geodesic curvature. We have $k_n(0) = 1$ for all $v \in T_pS$.*

**Example 2.35** *(**Skip this in class.**) Do Example 6 in p. 145. Here we use the fact from Lemma 2.17 to conclude that $dN_p = 0$, $p = (0,0,0)$. To be more precise, for each possible* ***normal section*** *$\alpha(s)$ of $S$ at $p$ in the direction $v$ we have curvature $k(0) = 0$ (due to the equation $z = y^4$) and so $k_n(0) = 0$, which gives*

$$-\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = k_n(0) = 0$$

*for all possible $\alpha'(0) \in T_pS$.*

**Example 2.36** ***(Curve lying on cylinder.)*** *Let $S$ be the cylinder in $\mathbb{R}^3$ given by*

$$\left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\right\} \tag{62}$$

*and we choose $N(x, y, z) = (-x, -y, 0)$. For $p \in S$, we can look at all possible* ***normal sections*** *of $S$ at $p$ in all possible directions $\pm v \in T_pS$ and conclude that, at $p \in S$, the maximal normal curvature is $+1$ and the minimal normal curvature is $0$.*

**Definition 2.37** *For $p \in S$, since the map $-dN_p : T_pS \to T_pS$ is self-adjoint, by Theorem 2.18 there exists orthonormal basis $\{e_1, e_2\}$ such that*

$$-dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \tag{63}$$

*for some $k_1,\ k_2 \in \mathbb{R}$ (without loss of generality we may assume $k_1 \geq k_2$), where $k_1,\ k_2$ satisfy*

$$\begin{cases} k_1 = \max_{v \in T_pS,\ |v|=1} \langle -dN_p(v), v\rangle = \textit{\textbf{maximal normal curvature}} \text{ at } p, \\ k_2 = \min_{v \in T_pS,\ |v|=1} \langle -dN_p(v), v\rangle = \textit{\textbf{minimal normal curvature}} \text{ at } p. \end{cases}$$

*We call them the* ***principal curvatures*** *of $S$ at $p$. The directions given by $\pm e_1,\ \pm e_2$ are called the* ***principal directions*** *of $S$ at $p$. Therefore, along any possible direction $\pm v \in T_pS$ at $p$, $|v| = 1$, the normal curvature $\langle -dN_p(\pm v), \pm v\rangle$ satisfies (now we denote $k_1,\ k_2$ at $p$ as $k_1(p)$ and $k_2(p)$)*

$$k_2(p) \leq \langle -dN_p(\pm v), \pm v\rangle \leq k_1(p), \quad \forall \text{ unit vector } v \in T_pS,$$

*i.e. $\langle -dN_p(\pm v), \pm v\rangle$ lies on the interval $[k_2(p), k_1(p)]$. We call it the* ***normal curvature interval*** *at $p \in S$.*

**Remark 2.38** ***(Be careful.)*** *In the above definition, we assume $\{e_1, e_2\}$ to be orthonormal even if $k_1 = k_2$. So by default,* ***principal directions*** *are* ***perpendicular*** *to each other. However, note that when $k_1 = k_2$ (denote it as $k$), we have $-dN_p = kI$ and any vector $v \neq 0 \in T_pS$ is an* ***eigenvector*** *and we usually say that the direction given by $v \neq 0 \in T_pS$ is a principal direction. This may be a little bit confusing sometimes !!!*

**Remark 2.39** ***Summary:*** *maximal and minimal normal curvatures are eigenvalues and principal directions are eigenvector directions, for the linear map $-dN$.*

**Lemma 2.40** *For each $p \in S$ and each number $\lambda \in [k_2(p), k_1(p)]$, there is some direction $\pm v_0 \in T_pS$ such that*

$$\langle -dN_p(\pm v_0),\ \pm v_0\rangle = \lambda.$$

**Remark 2.41** *We will see in Section 3.3 that if we choose $k_1(p) \geq k_2(p)$ for all $p \in S$, then $k_1(p)$ and $k_2(p)$ are continuous functions on $S$.*

**Proof.** This is obvious since the function $\langle -dN_p(\pm v), \pm v\rangle : v$ (unit vectors) $\in T_pS \to \mathbb{R}$ is continuous. Therefore, its image is a connected closed interval equal to $[k_2(p), k_1(p)]$. The result follows due to the intermediate value theorem. □

**Example 2.42** *Let $S = xy$-plane in $\mathbb{R}^3$. Then at any $p \in S$, all directions are principal directions and all normal curvatures at $p$ are $0$.*

**Example 2.43** *Let $S = S^2$ in $\mathbb{R}^3$ with $N(p) = -p$ (inward) on $S^2$. Then at any $p \in S$ all directions are principal directions and all normal curvatures at $p$ are $1$.*

**Example 2.44** *Let $S$ be the cylinder (62) in $\mathbb{R}^3$ with inward normal. At any $p \in S$, the directions parallel to the $z$-axis and the directions perpendicular to the $z$-axis are principal directions. The* ***principal curvatures*** *of $S$ at $p$ are $0$ and $1$.*

**Example 2.45** *Let $S$ be the hyperbolic paraboloid in Example 4 in p. 141. It has global parametrization $\mathbf{x}(u,v) = (u, v, v^2 - u^2)$, $(u,v) \in \mathbb{R}^2$, and we choose $N(u,v) = N(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(u,v)$. At $p = (0,0,0) = \mathbf{x}(0,0)$ we have $\mathbf{x}_u(0,0) = (1,0,0)$ and $\mathbf{x}_v(0,0) = (0,1,0)$, hence $T_pS = xy$-plane. We note that (see the computation in Example 2.12)*

$$-dN_p((1,0,0)) = (-2,0,0) = -2(1,0,0), \quad -dN_p((0,1,0)) = (0,2,0) = 2(0,1,0).$$

*Therefore, $-dN_p$ has two eigenvalues $-2$ and $2$ and corresponding eigenvectors $(1,0,0)$ and $(0,1,0)$. The* ***principal curvatures*** *of $S$ at $p$ are $-2$ and $2$. The two directions along the $x$-axis and $y$-axis are* ***principal directions****.*

### 2.1.4 Line of Curvature.

From now on, we always assume that $S \subset \mathbb{R}^3$ is a regular surface with a chosen orientation $N$.

**Definition 2.46** *Let $C \subset S$ be a connected regular curve with the property that the* ***tangent line*** *$L$ at any $p \in C$ is a* ***principal direction*** *of $S$ at $p$. Then we say $C$ is a* ***line of curvature*** *on $S$.*

**Remark 2.47** *Note that $C$ is in general a curve on $S$,* ***not*** *a straight line on $S$ (i.e. do not be misled by the name). On $S^2$, any curve $\alpha(s) \in S^2$ is a line of curvature. Similarly, on $\mathbb{R}^2$, any curve $\alpha(s) \in \mathbb{R}^2$ is a line of curvature.*

**Proposition 2.48** ***(This is Proposition 3 in p. 147.) (Olinde Rodrigues.)*** *A necessary and sufficiently for a connected regular $C$ on $S$ to be a* ***line of curvature*** *is that*

$$N'(t) = \lambda(t)\,\alpha'(t) \ \ (\textit{same as } -N'(t) = -\lambda(t)\,\alpha'(t)) \tag{64}$$

*for any parametrization $\alpha(t)$ of $C$ and any $t$ in the domain of $\alpha$, where $N(t) = N(\alpha(t))$ and $\lambda(t)$ is a differentiable function of $t$. In this case, $-\lambda(t)$ is the* ***principal curvature*** *of $S$ at $\alpha(t)$ along $\alpha'(t)$.*

**Remark 2.49** ***(Important.)*** *If a curve $\alpha(s) \in S$, $s \in I$, is a* ***line of curvature****, then each $\alpha'(s)$ is an eigenvector of $dN_{\alpha(s)}$ and by*

$$-\langle dN_p(\alpha'(s)), \alpha'(s)\rangle = -\langle N'(s), \alpha'(s)\rangle = \langle N(s), \alpha''(s)\rangle = k_n(s) = \textbf{\textit{normal curvature}} \text{ at } \alpha(s), \tag{65}$$

*we see that*

$$k_n(s) = k_1(s) \quad \textit{or} \quad k_2(s), \quad \forall\ s \in I. \tag{66}$$

**Proof.** For any parametrization $\alpha(t)$ of $C$, if it is a line of curvature, then $\alpha'(t)$ is a principal direction and we have

$$N'(t) = dN(\alpha'(t)) = \lambda(t)\,\alpha'(t)$$

for some function $\lambda(t)$. The function $\lambda(t)$ is **differentiable** due to the identity

$$\lambda(t) = \frac{\langle N'(t), \alpha'(t)\rangle}{\langle \alpha'(t), \alpha'(t)\rangle}, \quad \langle \alpha'(t), \alpha'(t)\rangle > 0 \text{ for all } t \in \text{domain of } \alpha.$$

Conversely, if we have the identity (64) for all $t$, it means that the vector $\alpha'(t) \neq 0$ is an eigenvector with eigenvalue $-\lambda(t)$ for the map $-dN_{\alpha(t)}$. Hence the curve $C$ is a line of curvature. ☐

### 2.1.5 Euler Formula, Gauss Curvature, Mean Curvature, and Umbilical Point.

Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation $N$. Let $\{e_1, e_2\}$ be an orthonormal basis on $T_pS$ (eigenvectors) corresponding to the two principal curvatures $k_1(p) \geq k_2(p)$. We also assume that they have positive orientation, i.e. they satisfy $e_1 \wedge e_2 = N(p)$ on $T_pS$. If $k_1(p) > k_2(p)$, such a basis on $T_pS$ is **unique** (**modulo the choice** $\{-e_1, -e_2\}$). For any unit vector $v \in T_pS$, one can express it as

$$v = (\cos\theta)\, e_1 + (\sin\theta)\, e_2, \quad |v| = 1,$$

where $\theta$ is the angle from $e_1$ to $v$ in the orientation of $T_pS$ (which means $e_1 \wedge v = N(p)$). The **normal curvature along** $v$ is now given by

$$\begin{aligned} k_n(p) &= II_p(v) = -\langle dN_p(v), v\rangle \\ &= -\langle dN_p((\cos\theta)\, e_1 + (\sin\theta)\, e_2),\ (\cos\theta)\, e_1 + (\sin\theta)\, e_2\rangle \\ &= \langle (k_1\cos\theta)\, e_1 + (k_2\sin\theta)\, e_2,\ (\cos\theta)\, e_1 + (\sin\theta)\, e_2\rangle = k_1(p)\cos^2\theta + k_2(p)\sin^2\theta. \end{aligned} \tag{67}$$

The formula in (67) is known as the **Euler formula**, which is the expression of the second fundamental form $II_p(\cdot)$ on $T_pS$ with respect to the basis $\{e_1, e_2\}$.

**Remark 2.50** *In Euler formula, it suffices to focus on* $\theta \in [0, \pi)$ *due to* $II_p(-v) = II_p(v)$.

**Remark 2.51** *(**Interesting observation.**) The formula (67) automatically implies the inequality*

$$k_2(p) \leq k_n(p) \leq k_1(p), \quad \forall\ v \in T_pS, \quad |v| = 1$$

*due to the linear combination*

$$k_n(p) = \rho k_1 + (1-\rho)\, k_2, \quad 0 \leq \rho = \cos^2\theta \leq 1. \tag{68}$$

**Definition 2.52** *Let* $k_1 \geq k_2$ *be the two **principal curvatures** of* $S$ *at* $p$*. The two numbers*

$$K := k_1 k_2, \qquad H := \frac{k_1 + k_2}{2} \tag{69}$$

*are called the **Gauss curvature** of* $S$ *at* $p$ *and the **mean curvature** of* $S$ *at* $p$ *respectively. We note that*

$$K = \det(-dN_p), \qquad H = \frac{1}{2}Tr(-dN_p) \ \left(= \frac{1}{2}Trace(-dN_p)\right). \tag{70}$$

**Remark 2.53** *If we change the orientation of* $S$ *(i.e. replace* $N$ *by* $-N$*), each principal curvature changes sign and so the mean curvature* $H$ *changes its sign. However, the Gauss curvature* $K$ *is unchanged (since* $\dim T_pS = 2$*).*

**Definition 2.54** *A point* $p \in S$ *is called*

1. ***Elliptic*** *if* $\det(dN_p) > 0$ *(i.e.* $k_1(p)$ *and* $k_2(p)$ *have the same sign).*
2. ***Hyperbolic*** *if* $\det(dN_p) < 0$ *(i.e.* $k_1(p)$ *and* $k_2(p)$ *have opposite sign).*
3. ***Parabolic*** *if* $\det(dN_p) = 0$*, but* $dN_p \neq 0$*. (i.e. either* $k_1(p)$ *or* $k_2(p)$ *is zero, but not both).*
4. ***Planar*** *if* $dN_p = 0$ *(i.e. both* $k_1(p)$ *and* $k_2(p)$ *are zero).*

**Remark 2.55** *Note that the above definition does not depend on the choice of the orientation* $N$ *on* $S$*.*

**Example 2.56** *(**See Example 5 in p. 142.**) For the graphic surface $S: z = x^2 + ky^2$ ($k > 0$ is a constant) with **upward normal**, the point $p = (0,0,0) \in S$ is an **elliptic point** with $k_1 = 2,\ k_2 = 2k$ and all normal curvatures have the same sign. All curves $\alpha(s) \in S,\ \alpha(0) = p$, passing through $p$ are bending towards the same side of $T_pS$ due to the constant sign of $\langle \alpha''(0),\ N(p)\rangle$.*

**Example 2.57** *For the graphic surface $S: z = y^2 - x^2$ with **upward normal**, the point $p = (0,0,0) \in S$ is a **hyperbolic point** with $k_1 = 2,\ k_2 = -2$. There are curves passing through $p$ bending towards one side of $T_pS$ and there are curves passing through $p$ bending towards the other side of $T_pS$.*

**Example 2.58** *Let $S$ be the cylinder in Example 3 in p. 141 with **inward normal**. At each $p \in S$, we have $k_1 = 1,\ k_2 = 0$. Therefore, all points on $S$ are **parabolic points**.*

**Definition 2.59** *Let $k_1 \geq k_2$ be the two **principal curvatures** of $S$ at $p$. If we have $k_1 = k_2$, then the point $p \in S$ is called an **umbilical point**. In particular, any **planar** point is an umbilical point.*

**Example 2.60** *On $S^2$, any point $p \in S^2$ is an umbilical point. Similarly, on $\mathbb{R}^2$, any point $p \in \mathbb{R}^2$ is an umbilical point.*

We have the following interesting result related to the above example:

**Proposition 2.61** *(**This is Proposition 4 in p. 149.**) Let $S \subset \mathbb{R}^3$ be a **connected** surface and all points on $S$ are **umbilical**. Then $S$ is contained either in a sphere (not necessarily unit sphere) or in a plane.*

**Proof.** Let $p \in S$ and $\mathbf{x}(u,v): U \subset \mathbb{R}^2 \to S$ a parametrization near $p$ and we take $U$ to be connected open set in $\mathbb{R}^2$. For each $q \in V = \mathbf{x}(U)$ and any vector $w = a\mathbf{x}_u + b\mathbf{x}_v \in T_qS$, we have

$$dN_q(w) = \lambda(q)\, w, \quad \forall\ w \in T_qS, \quad q \in V, \tag{71}$$

where $\lambda(q): V \to \mathbb{R}$ is a differentiable function on $V$ (you can see this from (72) below). Since $w = a\mathbf{x}_u + b\mathbf{x}_v$, we have

$$aN_u + bN_v = \lambda(q)(a\mathbf{x}_u + b\mathbf{x}_v).$$

Since $w \in T_qS$ is arbitrary, we have (pick $a = 1,\ b = 0$ and $a = 0,\ b = 1$ respectively)

$$N_u = \lambda(q)\,\mathbf{x}_u, \quad N_v = \lambda(q)\,\mathbf{x}_v, \quad \lambda(q) = \frac{\langle N_u, \mathbf{x}_u\rangle}{E(u,v)} = \frac{\langle N_u, \mathbf{x}_u\rangle}{G(u,v)}, \tag{72}$$

which gives (look at $N_{uv} - N_{vu} = 0$)

$$\lambda_v(q)\,\mathbf{x}_u - \lambda_u(q)\,\mathbf{x}_v = 0, \quad \forall\ q \in V$$

and so

$$\lambda_v(q) = \lambda_u(q) = 0, \quad \forall\ q \in V. \tag{73}$$

The above implies that $\lambda(q)$ is a **constant function** on $V$ since $V \subset S$ is connected.

**Case 1:** $\lambda(q) \equiv 0$ on $V$.

In this case, we have $N_u = N_v \equiv 0$ on $V$, which implies $N(u,v) = const.\ N_0$ on $V$. In particular we get

$$\frac{\partial}{\partial u}\langle \mathbf{x}(u,v),\ N_0\rangle = \frac{\partial}{\partial v}\langle \mathbf{x}(u,v),\ N_0\rangle = 0 \quad \text{on} \quad (u,v) \in U.$$

It means that all points $\mathbf{x}(u,v),\ (u,v) \in U$, lie on a plane $P$ perpendicular to $N_0$.

**Case 2:** $\lambda(q) \equiv \lambda \neq 0$ on $V$.

In this case, we have

$$N_u = \lambda \mathbf{x}_u, \quad N_v = \lambda \mathbf{x}_v, \quad \forall \ (u,v) \in U,$$

which gives

$$\left(\frac{N}{\lambda} - \mathbf{x}\right)_u = \left(\frac{N}{\lambda} - \mathbf{x}\right)_v = 0, \quad \forall \ (u,v) \in U,$$

i.e.

$$\mathbf{x}(u,v) - \frac{N(u,v)}{\lambda} = const.\ vector\ v_0 \quad \text{on} \quad U.$$

We conclude

$$|\mathbf{x}(u,v) - v_0|^2 = \frac{1}{\lambda^2} \quad \text{on} \quad U$$

and know that all points of $V$ are contained in a sphere with radius $1/|\lambda|$ centered at $v_0$.

Finally, for any $p,\ q \in S$, since $S$ is **connected**, there exists a continuous path $\alpha(t) \in S,\ t \in [0,1]$, such that $\alpha(0) = p,\ \alpha(1) = q$. By compactness of the set $\alpha([0,1])$, there exists finitely many connected coordinate neighborhoods $V_1,\ ...,\ V_k$ satisfying

$$\alpha([0,1]) \subset \bigcup_{i=1}^{k} V_i$$

and without loss of generality we may assume that

$$V_1 \bigcap V_2 \text{ (open set)} \neq \varnothing,\ V_2 \bigcap V_3 \text{ (open set)} \neq \varnothing,\ ...,\ V_{k-1} \bigcap V_k \text{ (open set)} \neq \varnothing.$$

and moreover, each coordinate neighborhoods $V_i$ is lying either on a plane or on a sphere with radius $r_i > 0$.

If $V_1$ lies on a plane, then $V_2$ also **lies on the same plane**. This is because if $V_2$ lies on a sphere or a different plane, then it is impossible for $V_1 \bigcap V_2$ to be an **open set** on the surface $S$. By induction, all $V_i,\ i = 3,\ 4,\ ...,\ k$, will all lie on the same plane. Hence $p$ and $q$ lie on the same plane. By fixing $p \in S$ and letting $q \in S$ be arbitrary, we see that the whole surface $S$ lies on some plane.

Similarly, if $V_1$ lies on a sphere with radius $r > 0$, the same argument implies that all $V_i,\ 1 \leq i \leq k$, lies on the same sphere with radius $r > 0$. Hence $p,\ q$ lie on the same sphere with radius $r > 0$. By fixing $p \in S$ and letting $q \in S$ be arbitrary, we see that the whole surface $S$ lies on a sphere with radius $r > 0$. The proof is done. □

### 2.1.6 Asymptotic Direction, Asymptotic Curve, Dupin Indicatrix, and Conjugate Directions.

**Definition 2.62** *Let $p \in S$. A direction $v \in T_pS$ is called an **asymptotic direction** at $p$ if the normal curvature of $S$ at $p$ along $v$ is $0$. A connected regular curve $C \subset S$ is called an **asymptotic curve** if for each $p \in C$ the **tangent line** $L$ at $p \in C$ is along an asymptotic **direction**.*

**Remark 2.63** *(**Comparison.**) If $\alpha(s) \in S,\ s \in I$, is a regular curve on $S$, we have:*

1. *If normal curvature $k_n(s)$ at $\alpha(s)$ along $\alpha'(s)$ is either $-k_1(s)$ or $-k_2(s)$ for all $s \in I$, then $\alpha(s)$ is **a line of curvature**.*

2. *If normal curvature $k_n(s)$ at $\alpha(s)$ along $\alpha'(s)$ is $0$ everywhere, then $\alpha(s)$ is **an asymptotic curve.***

**Lemma 2.64** *We have the following:*

1. *If $p \in S$ is an **elliptic** point ($k_1(p)\,k_2(p) > 0$), there is **no asymptotic direction** at $p$.*

2. *If $p \in S$ is a **hyperbolic** point ($k_1(p)\,k_2(p) < 0$), we have **exactly two asymptotic directions** at $p$. Moreover, if $k_1(p) + k_2(p) = 0$ (i.e. mean curvature is $0$), these two directions are **orthogonal**.*

3. *If $p \in S$ is a **parabolic** point ($k_1(p)\,k_2(p) = 0$, but one of them is nonzero), there is **exactly one asymptotic direction** at $p$.*

**Proof.** (1) is obvious.

For (2), let $\{e_1, e_2\}$ an orthonormal basis on $T_pS$ (eigenvectors of $-dN_p$) corresponding to the two principal curvatures $k_1(p) \geq k_2(p)$. For unit vector $v \in T_pS$ with $v = (\cos\theta)\,e_1 + (\sin\theta)\,e_2$, the **normal curvature** $k_n(p)$ **along** $v$ is given by

$$k_n(p) = k_1(p)\cos^2\theta + k_2(p)\sin^2\theta = [k_1(p) - k_2(p)]\cos^2\theta + k_2(p), \quad \theta \in [0, \pi), \tag{74}$$

where now $k_1(p) > 0$ and $k_2(p) < 0$. We want to find $\theta \in [0, \pi)$ such that $k_n(p) = 0$, i.e. want to solve the equation for $\theta \in [0, \pi)$ :

$$\cos^2\theta = -\frac{k_2(p)}{k_1(p) - k_2(p)} \in (0, 1),$$

which gives

$$\cos\theta = \pm\sqrt{-\frac{k_2(p)}{k_1(p) - k_2(p)}}. \tag{75}$$

The equation has exactly two solutions $\theta_0 \in (0, \pi/2)$ and $\pi - \theta_0 \in (\pi/2, \pi)$. Therefore, we have **exactly two** asymptotic directions at $p$ (draw a picture for this).

If $k_1(p) + k_2(p) = 0$, then equation (75) becomes $\cos\theta = \pm 1/\sqrt{2}$ and the two solutions are $\theta_0 = \pi/4,\ 3\pi/4$. These two directions are orthogonal.

For (3), if $k_1(p) > 0,\ k_2(p) = 0$, then equation $k_n(p) = 0$ becomes

$$k_1(p)\cos^2\theta = 0, \quad \theta \in [0, \pi).$$

The only solution is $\theta = \pi/2$ and there is exactly one asymptotic direction at $p$. On the other hand, if If $k_1(p) = 0,\ k_2(p) < 0$, then equation $k_n(p) = 0$ becomes

$$k_2(p)\sin^2\theta = 0, \quad \theta \in [0, \pi).$$

The only solution is $\theta = 0$ and there is exactly one asymptotic direction at $p$. □

**Definition 2.65** *Let $p \in S$. The **Dupin indicatrix** at $p$ is the set of vectors on $T_pS$ given by*

$$\{w \in T_pS : II_p(w) = -\langle dN_p(w), w\rangle = \pm 1\}. \tag{76}$$

Let $\{e_1, e_2\}$ an orthonormal basis on $T_pS$ (eigenvectors of $-dN_p$) corresponding to the two principal curvatures $k_1(p) \geq k_2(p)$. For $w \in T_pS$, we can express it as

$$w = xe_1 + ye_2 = (r\cos\theta)\,e_1 + (r\sin\theta)\,e_2, \quad r = |w|$$

and obtain

$$II_p(w) = -\langle dN_p(xe_1 + ye_2),\ xe_1 + ye_2\rangle = k_1x^2 + k_2y^2$$

or

$$\begin{aligned} II_p(w) &= -\langle dN_p((r\cos\theta)e_1 + (r\sin\theta)e_2), \ (r\cos\theta)e_1 + (r\sin\theta)e_2\rangle \\ &= r^2\left(k_1\cos^2\theta + k_2\sin^2\theta\right) = r^2 II_p(v), \quad \text{where} \quad v = \frac{w}{r}. \end{aligned}$$

Therefore, **the Dupin indicatrix is described by**

$$k_1x^2 + k_2y^2 = \pm 1, \tag{77}$$

which is a **symmetric quadratic curve** on the plane $T_pS$.

Similar to Lemma 2.64, we have:

**Lemma 2.66** *We have the following:*

1. *If $p \in S$ is an* ***elliptic*** *point ($k_1(p)k_2(p) > 0$), the Dupin indicatrix is an* ***ellipse*** *on $T_pS$. Moreover, if $k_1(p) = k_2(p)$, it is a* ***circle****.*
2. *If $p \in S$ is a* ***hyperbolic*** *point ($k_1(p)k_2(p) < 0$), the Dupin indicatrix is a* ***hyperbola*** *and* ***its two asymptotes are pointing to the two asymptotic directions*** *at $p$.*
3. *If $p \in S$ is a* ***parabolic*** *point ($k_1(p)k_2(p) = 0$, but one of them is nonzero), the Dupin indicatrix is a pair of* ***parallel lines*** *pointing to the* ***asymptotic direction*** *at $p$.*

**Proof.** (1) is obvious.

For (2), it suffices to verify the last statement. The hyperbola satisfies the equation

$$k_1(p)x^2 + k_2(p)y^2 = \pm 1, \quad k_1(p) > 0, \quad k_2(p) < 0.$$

We can decompose it as

$$\left(\sqrt{k_1(p)}x - \sqrt{-k_2(p)}y\right)\left(\sqrt{k_1(p)}x + \sqrt{-k_2(p)}y\right) = \pm 1$$

and its two asymptotes are the two lines $L_1, L_2$ given by

$$L_1: \sqrt{k_1(p)}x - \sqrt{-k_2(p)}y = 0, \qquad L_2: \sqrt{k_1(p)}x + \sqrt{-k_2(p)}y = 0.$$

The line $L_1$ intersects the unit circle on $T_pS$ at $(\cos\theta_0, \sin\theta_0)$ on the first quadrant $(+,+)$ at

$$\cos\theta_0 = \sqrt{-\frac{k_2(p)}{k_1(p) - k_2(p)}}, \quad \sin\theta_0 = \sqrt{\frac{k_1(p)}{k_1(p) - k_2(p)}} \tag{78}$$

and the line $L_2$ intersects the unit circle on $T_pS$ at $(\cos\theta_0, \sin\theta_0)$ on the second quadrant $(-,+)$ at

$$\cos\theta_0 = -\sqrt{-\frac{k_2(p)}{k_1(p) - k_2(p)}}, \quad \sin\theta_0 = \sqrt{\frac{k_1(p)}{k_1(p) - k_2(p)}}. \tag{79}$$

The above two directions are asymptotic directions due to (75).

For $(3)$, we may assume $k_1(p) > 0$, $k_2(p) = 0$. Now the Dupin indicatrix becomes

$$k_1(p)x^2 = \pm 1,$$

which is a pair of two parallel lines $x = \pm 1/\sqrt{k_1(p)}$ pointing in the $y$-axis direction (the direction with $\theta = \pi/2$, which is the only asymptotic direction at $p$). □

**Definition 2.67** *Let $p \in S$ and $w_1,\ w_2 \in T_pS$ are two **nonzero** vectors (may or may not be unit vectors). If we have*

$$\langle dN_p(w_1),\ w_2\rangle = \langle w_1,\ dN_p(w_2)\rangle = 0, \tag{80}$$

*we say these two vectors are **conjugate**. The two directions $r_1,\ r_2$ given by $w_1$ (or $-w_1$) and $w_2$ (or $-w_2$) are called **conjugate directions**.*

**Remark 2.68** *(**Be careful.**) Be careful that the definition (80) implies $dN_p(w_1) \perp w_2$ and $dN_p(w_2) \perp w_1$. However, in general, it **does not** imply that $dN_p(w_1) = \lambda_1 w_1$ and $dN_p(w_2) = \lambda_2 w_2$ for some constants $\lambda_1,\ \lambda_2$ (unless $w_1 \perp w_2$). The vector $dN_p(w_1)$ may have component in $w_2$ direction and the vector $dN_p(w_2)$ may have component in $w_1$ direction.*

**Example 2.69** *Assume at $p \in S$ we have $k_1 \neq k_2$ and **both are not** $0$. Then two principal directions are conjugate.*

**Example 2.70** *If $k_1 \neq k_2$ but one them is $0$, say $k_1 > 0$ (with unit eigenvector $e_1$), $k_2 = 0$ (with unit eigenvector $e_2,\ e_1 \perp e_2$), then for **any nonzero vector** $w \in T_pS$, the two vectors $w,\ e_2 \in T_pS$ are conjugate due to*

$$\langle dN_p(w),\ e_2\rangle = \langle w,\ dN_p(e_2)\rangle = \langle w,\ 0\rangle = 0. \tag{81}$$

**Example 2.71** *An asymptotic direction (normal curvature zero direction) is conjugate to itself.*

**Example 2.72** *If $k_1 = k_2 \neq 0$ (denote the common value as $\lambda,\ \lambda \neq 0$), any pair of **orthogonal directions** are conjugate. This is because $-dN_p = \lambda I$ and*

$$\langle dN_p(w_1),\ w_2\rangle = \langle -\lambda w_1,\ w_2\rangle = -\lambda\langle w_1,\ w_2\rangle = 0$$

*if and only if $\langle w_1,\ w_2\rangle = 0$. If $k_1 = k_2 = 0$, any pair of directions are conjugate.*

**Lemma 2.73** *(**Conjugate directions in terms of polar coordinates.**) Let $p \in S$ and $\{e_1, e_2\}$ is the orthonormal basis at $T_pS$ satisfying*

$$e_1 \wedge e_2 = N(p), \quad -dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \tag{82}$$

*where $k_1 \neq k_2$ are principal curvatures. Let $r_1,\ r_2$ be two directions and $\theta,\ \varphi$ are the angles from $e_1$ to $r_1$, $r_2$ respectively in the orientation of $T_pS$. Then $r_1,\ r_2$ are **conjugate** if and only if*

$$k_1 \cos\theta\cos\varphi + k_2 \sin\theta\sin\varphi = 0. \tag{83}$$

**Remark 2.74** *If we replace $\theta$ by $\theta + \pi$ (or $\varphi$ by $\varphi + \pi$), the identity (83) still holds.*

**Proof.** By the assumption, $r_1,\ r_2$ are conjugate if and only if the two vectors

$$w_1 = (\cos\theta)\, e_1 + (\sin\theta)\, e_2, \quad w_2 = (\cos\varphi)\, e_1 + (\sin\varphi)\, e_2 \tag{84}$$

are conjugate. We compute

$$\begin{aligned} &\langle dN_p(w_1),\ w_2\rangle \\ &= \langle dN_p((\cos\theta)\, e_1 + (\sin\theta)\, e_2),\ (\cos\varphi)\, e_1 + (\sin\varphi)\, e_2\rangle \\ &= -k_1\cos\theta\cos\varphi - k_2\sin\theta\sin\varphi \end{aligned}$$

and so $\langle dN_p(w_1),\ w_2\rangle = 0$ if and only if (83) holds. □

**Lemma 2.75** *(**Conjugate directions in terms of Euclidean coordinates.**) Same assumption as in Lemma 2.73 with $k_1 \neq k_2$. Assume the direction $r_1$ is given by some **nonzero** vector $w_1 = xe_1+ye_2 \in T_pS$ and the direction $r_2$ is given by another **nonzero** vector $w_2 = \tilde{x}e_1+\tilde{y}e_2 \in T_pS$. Then $r_1,\ r_2$ are **conjugate** if and only if*

$$k_1 x\tilde{x} + k_2 y\tilde{y} = 0. \tag{85}$$

*In particular, for any **nonzero** vector $w_1 = xe_1 + ye_2 \in T_pS$, if*

$$w_2 = (-k_2 y)\, e_1 + (k_1 x)\, e_2 \in T_pS \tag{86}$$

*is a **nonzero vector,** then $w_1$ and $w_2$ are conjugate.*

**Remark 2.76** *The above lemma is still correct if $k_1 = k_2 \neq 0$. In case $k_1 > 0,\ k_2 = 0$, (85) becomes $k_1 x\tilde{x} = 0$. In such a case, any nonzero $(0, y)$ and any nonzero $(\tilde{x}, \tilde{y})$ are conjugate. See Example 2.69.*

**Proof.** We can write $w_1 = \sqrt{x^2+y^2}\,(\cos\theta, \sin\theta)$ and $w_2 = \sqrt{\tilde{x}^2+\tilde{y}^2}\,(\cos\varphi, \sin\varphi)$. By (83), they are conjugate if and only if

$$k_1 \cos\theta\cos\varphi + k_2 \sin\theta\sin\varphi = 0,$$

which is the same as

$$k_1\left(\sqrt{x^2+y^2}\cos\theta\right)\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\cos\varphi\right) + k_2\left(\sqrt{x^2+y^2}\sin\theta\right)\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\sin\varphi\right) = 0,$$

i.e. if and only if (85) holds. In particular, the two nonzero vectors $(x, y)$, $(-k_2 y, k_1 x)$ are conjugate.□

**Example 2.77** *(**Using Dupin indicatrix to find conjugate directions.**) Using the property that any two **nonzero** vectors $w_1 = (x, y)$, $w_2 = (-k_2 y, k_1 x)$ are conjugate to each other, one can explain the picture in the textbook, p. 152 for the elliptic case. The construction is also valid for the hyperbolic case. **Draw two pictures on blackboard** (one for elliptic case and one for hyperbolic case).*

### 2.1.7 Conclusion for Conjugate Directions.

Let $p \in S$ and $\{e_1, e_2\}$ is the orthonormal basis at $T_pS$ satisfying

$$e_1 \wedge e_2 = N(p), \quad -dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \tag{87}$$

where $k_1,\ k_2$ are principal curvatures. Let $r_1$ be a direction on $T_pS$ given by $w_1 = (\cos\theta)\, e_1 + (\sin\theta)\, e_2$. By Lemma 2.73, we can conclude the following:

(1). Assume $k_1 \neq k_2$ and $k_1 k_2 > 0$ (i.e. $p \in S$ is an **elliptic** point) (without loss of generality, we may assume $k_1 > 0$ and $k_2 > 0$). In such a case the vector $(k_1\cos\theta, k_2\sin\theta) \neq (0,0)$ is nonzero and there is **an unique conjugate direction** $r_2$ given by (note that $-w_2$ also gives rise to the **same** direction)

$$w_2 = (\cos\varphi)\, e_1 + (\sin\varphi)\, e_2,$$

where the direction $(\cos\varphi, \sin\varphi)$ is perpendicular to the direction $(k_1\cos\theta, k_2\sin\theta)$, i.e.

$$k_1\cos\theta\cos\varphi + k_2\sin\theta\sin\varphi = 0 \text{ (same as } \begin{pmatrix} k_1\cos\theta \\ k_2\sin\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = 0). \tag{88}$$

Moreover, the direction $r_2$ is **different from** $r_1$ (due to $k_1\cos^2\theta + k_2\sin^2\theta > 0$). In particular, if $r_1$ is the **principal direction** $e_1$ (same as $\theta = 0$), then the identity (88) is the same as $\cos\varphi = 0$ (i.e. $\varphi = \frac{\pi}{2}$) and we have $r_2$ is the **principal direction** $e_2$. Similarly, if $r_1$ is the **principal direction** $e_2$, then $r_2$ is the **principal direction** $e_1$.

(2) . If $k_1 \neq k_2$ and $k_1 k_2 < 0$ (i.e. $p \in S$ is a **hyperbolic** point), then we have the same conclusion as in (1) , but the direction $r_2$ may be the same as $r_1$ if we have

$$k_1 \cos^2 \theta + k_2 \sin^2 \theta = 0 \text{ (note that } k_1 > 0,\ k_2 < 0),$$

which means that $r_1$ is an **asymptotic direction**. Therefore, as long as $r_1$ is **not an asymptotic direction**, the conjugate direction $r_2$ is **unique and different from** $r_1$. **But if** $r_1$ **is an asymptotic direction, the only direction conjugate to** $r_1$ **is** $r_1$ **itself.** Finally, if $r_1$ is the **principal direction** $e_1$ $(e_2)$, then $r_2$ is the **principal direction** $e_2$ $(e_1)$.

(3) . If $k_1 \neq k_2$ and $k_1 k_2 = 0$ (i.e. $p \in S$ is a **parabolic** point) (without loss of generality, we may assume $k_1 > 0,\ k_2 = 0$), then the identity (88) is the same as $\cos\theta\cos\varphi = 0$. Therefore, if $\theta = \frac{\pi}{2}$ (same as $w_1 = e_2$, eigenvector for eigenvalue $k_2 = 0$), then **any nonzero vector** $w_2 \in T_pS$ is conjugate to $r_1$. But if $\theta \neq \frac{\pi}{2}$ (same as $w_1 \neq e_2$), then we have $\cos\varphi = 0$ and **the only direction** conjugate to $r_1$ is the eigenvector direction $e_2$.

Therefore, for a parabolic point, we have the property: **if two nonzero vectors** $w_1,\ w_2 \in T_pS$ **are conjugate to each other, then one of them must be pointing to the eigenvector direction** $e_2$, **where** $k_2 = 0$.

(4) . If $k_1 = k_2 = \lambda \neq 0$, then we have $-dN_p = \lambda I,\ \lambda \neq 0$, and for any nonzero vectors $w_1,\ w_2 \in T_pS$, we have

$$\langle dN_p (w_1),\ w_2 \rangle = \langle -\lambda w_1,\ w_2 \rangle = -\lambda \langle w_1,\ w_2 \rangle .$$

Therefore, $w_1,\ w_2 \in T_pS$ are conjugate if and only if $\langle w_1,\ w_2 \rangle = 0$.

(5) . If $k_1 = k_2 = \lambda = 0$, then $-dN_p = 0$ and any pair of directions are conjugate.

### 2.1.8 Three Interesting Identities Related to Gauss Curvature and Mean Curvature.

**Lemma 2.78** *Let* $p \in S$ *and* $\{e_1, e_2\}$ *be the orthonormal basis at* $T_pS$ *satisfying*

$$e_1 \wedge e_2 = N(p), \quad -dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \tag{89}$$

*where* $k_1,\ k_2$ *are principal curvatures. We have:*

1. If $H(p) = 0$ with $k_1(p) > 0,\ k_2(p) < 0$, we have the identity

$$\langle dN_p(w_1),\ dN_p(w_2) \rangle = -K(p) \langle w_1, w_2 \rangle, \quad \forall\ w_1,\ w_2 \in T_p(S), \tag{90}$$

where $K(p) < 0$ is the Gauss curvature at $p$. In particular, if we assume $H \equiv 0$ on $S$ with $k_1 > 0,\ k_2 < 0$ everywhere, then the angle of two intersecting curves on $S$ and the angle of their spherical images are equal **up to a sign** (in such a case, for any $p \in S$, $dN_p$ is a **conformal** map because it preserves angles). □

2. We have

$$dN_p(w_1) \wedge dN_p(w_2) = K(p)(w_1 \wedge w_2), \quad \forall\ w_1,\ w_2 \in T_p(S). \tag{91}$$

3. We have

$$dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2) = -2H(p)(w_1 \wedge w_2), \quad \forall\ w_1,\ w_2 \in T_p(S). \tag{92}$$

**Remark 2.79** *In the case when we have $w_1 \wedge w_2 = N$, then (91) and (92) give the following*

$$K(p) = \langle dN_p(w_1) \wedge dN_p(w_2), \ N(p) \rangle \tag{93}$$

*and*

$$-2H(p) = \langle dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2), \ N(p) \rangle . \tag{94}$$

**Proof.** We prove (2) and (3) first. For any $w_1, \ w_2 \in T_p(S)$, we can express them as

$$w_1 = a_1 e_1 + a_2 e_2, \quad w_2 = b_1 e_1 + b_2 e_2 \in T_p(S)$$

for some constants $a_1, \ a_2, \ b_1, \ b_2$. We have

$$w_1 \wedge w_2 = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2$$

and

$$\begin{aligned}
& dN_p(w_1) \wedge dN_p(w_2) \\
& = (-a_1 k_1 e_1 - a_2 k_2 e_2) \wedge (-b_1 k_1 e_1 - b_2 k_2 e_2) \\
& = (a_1 k_1 e_1 + a_2 k_2 e_2) \wedge (b_1 k_1 e_1 + b_2 k_2 e_2) = (a_1 k_1 b_2 k_2 - a_2 k_2 b_1 k_1) e_1 \wedge e_2 \\
& = k_1 k_2 (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 = K(p)(w_1 \wedge w_2), \quad \forall \ w_1, \ w_2 \in T_p(S).
\end{aligned}$$

We also have

$$\begin{aligned}
& dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2) \\
& = (-a_1 k_1 e_1 - a_2 k_2 e_2) \wedge (b_1 e_1 + b_2 e_2) + (a_1 e_1 + a_2 e_2) \wedge (-b_1 k_1 e_1 - b_2 k_2 e_2) \\
& = [-a_1 k_1 b_2 + a_2 k_2 b_1 + a_1(-b_2 k_2) + a_2 b_1 k_1] e_1 \wedge e_2 \\
& = [a_1 b_2 (-k_1 - k_2) + a_2 b_1 (k_1 + k_2)] e_1 \wedge e_2 = -2H(p)(a_1 b_2 - a_2 b_1) e_1 \wedge e_2 \\
& = -2H(p)(w_1 \wedge w_2), \quad \forall \ w_1, \ w_2 \in T_p(S).
\end{aligned}$$

For (1), we first have

$$K(p) = k_1(p) k_2(p) = -k_1^2(p) = -k_2^2(p)$$

and

$$\begin{aligned}
& \langle dN_p(w_1), \ dN_p(w_2) \rangle = \langle -a_1 k_1 e_1 - a_2 k_2 e_2, \ -b_1 k_1 e_1 - b_2 k_2 e_2 \rangle \\
& = a_1 b_1 k_1^2 + a_2 b_2 k_2^2 = -(a_1 b_1 + a_2 b_2) K(p) = -K(p) \langle w_1, w_2 \rangle, \quad \forall \ w_1, \ w_2 \in T_p(S),
\end{aligned}$$

where $K(p) < 0$. In particular, we have

$$|dN_p(w)|^2 = -K(p)|w|^2 \ \text{(same as } |dN_p(w)| = \sqrt{-K(p)}|w| \text{)}, \quad \forall \ w \in T_p S. \tag{95}$$

If we assume $H \equiv 0$ on $S$ with $k_1 > 0, \ k_2 < 0$ everywhere, then for any two regular curves $\alpha(s), \ \beta(s)$ on $S$ with intersection point at $\alpha(0) = \beta(0) = q \in S$ with angle $\theta$, we have

$$\langle dN_q(\alpha'(0)), \ dN_q(\beta'(0)) \rangle = -K(q) \langle \alpha'(0), \ \beta'(0) \rangle ,$$

where we know that

$$dN_q(\alpha'(0)) = \left. \frac{d}{ds} \right|_{s=0} N(\alpha(s)), \quad dN_q(\beta'(0)) = \left. \frac{d}{ds} \right|_{s=0} N(\beta(s)).$$

The intersection angle $\theta_N$ of the two spherical image curves $N(\alpha(s)), \ N(\beta(s))$ at $N(q)$ satisfies

$$\begin{aligned}
\cos\theta_N & = \frac{\langle dN_q(\alpha'(0)), \ dN_q(\beta'(0)) \rangle}{|dN_q(\alpha'(0))| \, |dN_q(\beta'(0))|} \\
& = \frac{-K(q) \langle \alpha'(0), \ \beta'(0) \rangle}{\sqrt{-K(q)} |\alpha'(0)| \sqrt{-K(q)} |\beta'(0)|} = \frac{\langle \alpha'(0), \ \beta'(0) \rangle}{|\alpha'(0)| \, |\beta'(0)|} = \cos\theta,
\end{aligned}$$

which implies $\theta_N = \pm\theta$. The proof is done. $\square$

### 2.1.9 The Geodesic Torsion for a Regular Curve on a Regular Surface $S \subset \mathbb{R}^3$ (this is Exercise 19 in p. 155).

Let $\alpha(s),\ s \in I$, be a regular curve on $S$. In the following, we shall compare the **Frenet frame** equations for the moving frame $\{t(s),\ n(s),\ b(s)\}$ along $\alpha(s) \in \mathbb{R}^3$ and the differential equations for the **surface moving frame** $\{t(s),\ \mathbf{n}_{int}(s),\ N(s)\}$ along $\alpha(s) \in S$. Recall that $\mathbf{n}_{int}(s)$ is the **intrinsic normal** of $\alpha$ at $s$, defined as

$$\mathbf{n}_{int}(s) = N(s) \wedge t(s) = N(s) \wedge \alpha'(s), \quad s \in I \quad (N(s) \text{ means } N(\alpha(s))). \tag{96}$$

By (46), we have

$$\begin{aligned} t'(s) &= \underbrace{\alpha''(s)} = k(s)\, n(s) = \underbrace{\langle \alpha''(s),\ \mathbf{n}_{int}(s)\rangle}\, \mathbf{n}_{int}(s) + \underbrace{\langle \alpha''(s),\ N(s)\rangle}\, N(s) \\ &= \underbrace{k_g(s)\, \mathbf{n}_{int}(s) + k_n(s)\, N(s)}, \quad s \in I, \end{aligned} \tag{97}$$

where $k_n(s)$ is the **geodesic curvature** of $\alpha$ and $k_n(s)$ is the **normal curvature** of $\alpha$.

To continue, we would like to express $\{t(s),\ n(s),\ b(s)\}$ in terms of $\{t(s),\ \mathbf{n}_{int}(s),\ N(s)\}$. The following lemma is straightforward.

**Lemma 2.80** *We have the following expression:*

$$\begin{cases} n(s) = \underbrace{\langle n(s),\ \mathbf{n}_{int}(s)\rangle}\, \mathbf{n}_{int}(s) + \underbrace{\langle n(s),\ N(s)\rangle}\, N(s) \\ b(s) = -\underbrace{\langle n(s),\ N(s)\rangle}\, \mathbf{n}_{int}(s) + \underbrace{\langle n(s), \mathbf{n}_{int}(s)\rangle}\, N(s). \end{cases} \tag{98}$$

*In terms of formal matrix notation we have*

$$\begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle n(s),\ \mathbf{n}_{int}(s)\rangle & \langle n(s),\ N(s)\rangle \\ 0 & -\langle n(s),\ N(s)\rangle & \langle n(s),\ \mathbf{n}_{int}(s)\rangle \end{pmatrix} \begin{pmatrix} t(s) \\ \mathbf{n}_{int}(s) \\ N(s) \end{pmatrix}, \quad s \in I. \tag{99}$$

*where the coefficient matrix is* ***orthogonal****.*

**Proof.** The first identity in (98) is clear. For the second identity in (98), we have

$$\begin{aligned} b(s) &= t(s) \wedge n(s) = t(s) \wedge \left[\underbrace{\langle n(s), \mathbf{n}_{int}(s)\rangle}\, \mathbf{n}_{int}(s) + \underbrace{\langle n(s),\ N(s)\rangle}\, N(s)\right] \\ &= \underbrace{\langle n(s), \mathbf{n}_{int}(s)\rangle}\, N(s) - \underbrace{\langle n(s),\ N(s)\rangle}\, \mathbf{n}_{int}(s). \end{aligned}$$

The proof is done. □

Next, we compute $N'(s)$ (note that $N(s)$ means $N(\alpha(s))$) and can express it as

$$N'(s) = A(s)\, t(s) + Q(s)\, \mathbf{n}_{int}(s) + C(s)\, N(s), \quad s \in I$$

for some coefficients $A(s),\ Q(s),\ C(s)$ and we see that $C(s) \equiv 0$ (since $\langle N'(s), N(s)\rangle = 0$ for all $s$) and

$$A(s) = \langle N'(s), t(s)\rangle = \langle N'(s), \alpha'(s)\rangle = -\langle N(s), \alpha''(s)\rangle = -k_n(s),$$

which is the normal curvature. Hence we conclude (we will give $Q(s)$ a name later on)

$$N'(s) = -k_n(s)\, t(s) + Q(s)\, \mathbf{n}_{int}(s), \quad s \in I. \tag{100}$$

Finally, we compute

$$\begin{aligned}
\mathbf{n}_{int}'(s) &= \frac{d}{ds}\left[N(s)\wedge\alpha'(s)\right] = N'(s)\wedge\alpha'(s) + N(s)\wedge\alpha''(s)\\
&= \left[-k_n(s)\,t(s) + Q(s)\,\mathbf{n}_{int}(s)\right]\wedge t(s) + N(s)\wedge\left[k_g(s)\,\mathbf{n}_{int}(s) + k_n(s)\,N(s)\right]\\
&= -k_g(s)\,t(s) - Q(s)\,N(s), \quad s\in I.
\end{aligned}$$

We define the following:

**Definition 2.81** *(See textbook p. 155.) The quantity $Q(s)$ is called the* ***geodesic torsion*** *of $\alpha$ at $\alpha(s)$ and denote it as $\tau_g(s)$. Note that we can also express it as*

$$\tau_g(s) = \langle N'(s),\ \mathbf{n}_{int}(s)\rangle = -\langle N(s),\ \mathbf{n}_{int}'(s)\rangle, \quad s\in I. \tag{101}$$

**Remark 2.82** ***(Important.)*** *The geodesic torsion $\tau_g(s)$ is a* ***new quantity****. It* ***cannot be expressed*** *in terms of $k_n(s)$ and $k_g(s)$. Note that if we do the following:*

$$\begin{aligned}
\tau_g(s) &= \langle N'(s),\ \mathbf{n}_{int}(s)\rangle = -\langle N(s),\ \mathbf{n}_{int}'(s)\rangle = -\left\langle N(s),\ \frac{d}{ds}\left(N(s)\wedge t(s)\right)\right\rangle\\
&= -\langle N(s),\ N'(s)\wedge t(s) + N(s)\wedge t'(s)\rangle = -\langle N(s),\ N'(s)\wedge t(s)\rangle\\
&= -\langle N(s),\ \left[-k_n(s)\,t(s) + \tau_g(s)\,\mathbf{n}_{int}(s)\right]\wedge t(s)\rangle = -\langle N(s),\ \tau_g(s)\,\mathbf{n}_{int}(s)\wedge t(s)\rangle = \tau_g(s),
\end{aligned}$$

*we get nothing useful at all.*

We can summarize the following:

**Lemma 2.83** ***(Moving frame equations on*** *$S$.) The* ***surface moving frame*** *$\{t(s),\ \mathbf{n}_{int}(s),\ N(s)\}$ along $\alpha(s)\in S$ (with normal $N(s)$) satisfies the equation*

$$\begin{pmatrix} t'(s)\\ \mathbf{n}_{int}'(s)\\ N'(s)\end{pmatrix} = \begin{pmatrix} 0 & k_g(s) & k_n(s)\\ -k_g(s) & 0 & -\tau_g(s)\\ -k_n(s) & \tau_g(s) & 0\end{pmatrix}\begin{pmatrix} t(s)\\ \mathbf{n}_{int}(s)\\ N(s)\end{pmatrix}, \quad s\in I. \tag{102}$$

**Remark 2.84** *Compare with the Frenet frame equations for $\alpha(s)\in\mathbb{R}^3$, given by*

$$\begin{pmatrix} t'(s)\\ n'(s)\\ b'(s)\end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0\\ -k(s) & 0 & -\tau(s)\\ 0 & \tau(s) & 0\end{pmatrix}\begin{pmatrix} t(s)\\ n(s)\\ b(s)\end{pmatrix}, \quad \begin{pmatrix} t(s)\\ n(s)\\ b(s)\end{pmatrix}\in\mathbb{R}^9. \tag{103}$$

The following lemma is about the relation between **geodesic torsion** and **principal curvatures** of $S$:

**Lemma 2.85** *Let $p\in S$ and $\alpha(s)\in S,\ s\in I$, is a regular curve with $\alpha(0)=p$ and let $\{e_1, e_2\}$ be the orthonormal basis at $T_pS$ satisfying*

$$e_1\wedge e_2 = N(p), \quad -dN_p(e_1) = k_1e_1, \quad -dN_p(e_2) = k_2e_2. \tag{104}$$

*Denote the angle from $e_1$ to $t(0)=\alpha'(0)$ by $\varphi$. Then we have*

$$\tau_g(0) = (k_1 - k_2)\cos\varphi\sin\varphi. \tag{105}$$

**Proof.** By (102), we have (note that $N(0)\wedge e_1 = e_2,\ N(0)\wedge e_2 = -e_1$)

$$\begin{aligned}\tau_g(0) &= \langle N'(0),\ \mathbf{n}_{int}(0)\rangle = \langle N'(0),\ N(0)\wedge t(0)\rangle \\ &= \left\langle N'(0),\ N(0)\wedge\left[\underbrace{(\cos\varphi)e_1 + (\sin\varphi)e_2}\right]\right\rangle = \langle N'(0),\ (\cos\varphi)e_2 - (\sin\varphi)e_1\rangle \\ &= (\cos\varphi)\langle N'(0),\ e_2\rangle - (\sin\varphi)\langle N'(0),\ e_1\rangle.\end{aligned}$$

Now we note that

$$\begin{aligned}N'(0) &= dN_p(\alpha'(0)) = dN_p\left(\underbrace{(\cos\varphi)e_1 + (\sin\varphi)e_2}\right) \\ &= (\cos\varphi)(-k_1e_1) + (\sin\varphi)(-k_2e_2),\end{aligned}$$

which gives

$$\begin{aligned}\tau_g(0) &= (\cos\varphi)\langle N'(0),\ e_2\rangle - (\sin\varphi)\langle N'(0),\ e_1\rangle \\ &= (\cos\varphi)(\sin\varphi)(-k_2) - (\sin\varphi)(\cos\varphi)(-k_1) = (k_1 - k_2)(\cos\varphi)(\sin\varphi).\end{aligned}$$

□

**Lemma 2.86** ***(Relation between torsion and geodesic torsion.)*** *Let $p\in S$ and $\alpha(s)\in S,\ s\in I$, is a regular curve with $\alpha(0) = p$. By (99), we have*

$$n(s) = \langle n(s),\ \mathbf{n}_{int}(s)\rangle\,\mathbf{n}_{int}(s) + \langle n(s),\ N(s)\rangle\,N(s),\quad s\in I$$

*and if we write it as*

$$n(s) = \underbrace{\langle n(s),\ N(s)\rangle}\,N(s) + \underbrace{\langle n(s),\ \mathbf{n}_{int}(s)\rangle}\,\mathbf{n}_{int}(s),\quad s\in I$$

*and let (i.e. the angle between $n(s)$ and $N(s)$ is denoted as $\theta(s)$)*

$$\langle n(s),\ N(s)\rangle = \cos\theta(s),\quad \langle n(s),\ \mathbf{n}_{int}(s)\rangle = \sin\theta(s),\quad s\in I, \tag{106}$$

*then we have*

$$\theta'(s) = \tau(s) - \tau_g(s),\quad s\in I, \tag{107}$$

*where $\tau(s)$ is the torsion of $\alpha(s)$ as a curve in $\mathbb{R}^3$.*

**Remark 2.87** *If we let $\langle n(s),\ N(s)\rangle = \sin\theta(s)$, then $\theta'(s) = \tau_g(s) - \tau(s),\ s\in I$.*

**Proof.** By (102) and (103), we have

$$\begin{aligned}&(-\sin\theta(s))\,\theta'(s) = \frac{d}{ds}\langle n(s),\ N(s)\rangle = \langle n'(s),\ N(s)\rangle + \langle n(s),\ N'(s)\rangle \\ &= \langle -k(s)t(s) - \tau(s)b(s),\ N(s)\rangle + \langle n(s),\ -k_n(s)t(s) + \tau_g(s)\mathbf{n}_{int}(s)\rangle \\ &= -\tau(s)\langle b(s),\ N(s)\rangle + \tau_g(s)\langle n(s),\ \mathbf{n}_{int}(s)\rangle \\ &= -\tau(s)\left\langle -\underbrace{\langle n(s),\ N(s)\rangle}\mathbf{n}_{int}(s) + \underbrace{\langle n(s),\mathbf{n}_{int}(s)\rangle}N(s),\ N(s)\right\rangle + \tau_g(s)\langle n(s),\ \mathbf{n}_{int}(s)\rangle \\ &= \underbrace{\langle n(s),\mathbf{n}_{int}(s)\rangle}(\tau_g(s) - \tau(s)),\quad \text{where}\quad \langle n(s),\mathbf{n}_{int}(s)\rangle = \sin\theta(s).\end{aligned}$$

The proof is done. □

**Lemma 2.88** ***(Lines of curvature have zero geodesic torsion.)*** *Let $p\in S$ and $\alpha(s)\in S,\ s\in I$, is a regular curve with $\alpha(0) = p$. Then $\alpha(s)$ is a* ***line of curvature*** *if and only if $\tau_g(s)\equiv 0$ along the whole curve $\alpha(s),\ s\in I$.*

**Proof.** ($\Longrightarrow$) Assume $\alpha(s)$ is a line of curvature. We have

$$N'(s) = dN_{\alpha(s)}(\alpha'(s)) = \lambda(s)\alpha'(s), \quad \forall\, s \in I$$

for some function $\lambda(s)$ and by (102) we also have

$$N'(s) = -k_n(s)t(s) + \tau_g(s)\mathbf{n}_{int}(s), \quad s \in I. \tag{108}$$

Comparing the above two identities, we have $\tau_g(s) \equiv 0$ along the whole curve $\alpha(s)$, $s \in I$.

($\Longleftarrow$) Assume $\tau_g(s) \equiv 0$ for all $s \in I$. By the identity (108), we must have $N'(s) = -k_n(s)t(s)$ for all $s \in I$, which means that $\alpha(s)$ is a line of curvature. □

## 2.2 The Gauss Map in Local Coordinates (this is Section 3-3 of the book).

The purpose of this section is to use local coordinates to study the second fundamental form and the differential of Gauss map. Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation $N$. All local parametrizations $\mathbf{x}(u,v)$ of $S$ in this section are assumed to be compatible with the orientation $N$ of $S$. That is

$$N(u,v) = N(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$$

for all $(u,v)$ in the domain of $\mathbf{x}$.

Let $\mathbf{x}(u,v)$ be a parametrization near $p \in S$. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a curve in $S$ with $\alpha(0) = p$. In the following all computations are evaluated at the point $p$ and at $t = 0$ unless otherwise stated. We have

$$dN(\alpha') = dN(\mathbf{x}_u u' + \mathbf{x}_v v') = \frac{d}{dt}N(u,v) = N_u u' + N_v v',$$

where $N(u,v)$ means $N(\mathbf{x}(u,v))$ and we know that $dN(\mathbf{x}_u) = N_u,\ dN(\mathbf{x}_v) = N_v$.

Since $N_u,\ N_v \in T_pS$, we can write

$$\begin{cases} N_u = dN(\mathbf{x}_u) = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\ N_v = dN(\mathbf{x}_v) = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \end{cases} \tag{109}$$

for some constants $a_{ij},\ 1 \le i,\ j \le 2$. The above means that the matrix representation $M$ for the linear map $dN_p : T_pS \to T_pS$, with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, is given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{110}$$

In particular, we have

$$\begin{aligned} dN(\alpha') &= dN(\mathbf{x}_u u' + \mathbf{x}_v v') = (a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v)u' + (a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v)v' \\ &= (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v, \end{aligned}$$

which, in terms of matrix formulation, is

$$M\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}. \tag{111}$$

**Remark 2.89** *Note that the matrix representation $M$ for $dN_p$ with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is* ***not necessarily symmetric*** *even if $dN_p$ is* ***self-adjoint****. The matrix $M$ is symmetric if $\{\mathbf{x}_u, \mathbf{x}_v\}$ is orthonormal due to*

$$\begin{cases} \langle dN(\mathbf{x}_u), \mathbf{x}_v\rangle = \langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_v\rangle = a_{21}, \\ \langle dN(\mathbf{x}_v), \mathbf{x}_u\rangle = \langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_u\rangle = a_{12}, \\ \langle dN(\mathbf{x}_u), \mathbf{x}_v\rangle = \langle dN(\mathbf{x}_v), \mathbf{x}_u\rangle . \end{cases}$$

*Now we look at an example: let $T(x,y) = (2x, -3y) : \mathbb{R}^2 \to \mathbb{R}^2$ be a* ***self-adjoint*** *linear map and choose the* ***non-orthonormal*** *basis $\{v_1, v_2\} = \{(1,0),\ (1,1)\}$. We have*

$$Tv_1 = 2v_1, \quad Tv_2 = 5v_1 - 3v_2.$$

*Therefore, the matrix representation for $T : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to $\{v_1, v_2\}$ is*

$$M = \begin{pmatrix} 2 & 5 \\ 0 & -3 \end{pmatrix},$$

*which is not symmetric.*

Now we compare two different ways to express $II_p(\alpha')$, where $\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'$.

**First way:**

We have

$$\begin{aligned} II_p(\alpha') &= -\langle dN(\alpha'), \alpha'\rangle = -\left\langle \underbrace{N_u u' + N_v v'}, \ \mathbf{x}_u u' + \mathbf{x}_v v' \right\rangle \\ &= \boxed{(u', v')\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}} = e(u')^2 + 2fu'v' + g(v')^2, \end{aligned} \tag{112}$$

where the quantities $e,\ f,\ g$ are given by

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} -\langle N_u,\ \mathbf{x}_u\rangle & -\langle N_u,\ \mathbf{x}_v\rangle \\ -\langle N_v,\ \mathbf{x}_u\rangle & -\langle N_v,\ \mathbf{x}_v\rangle \end{pmatrix} = \begin{pmatrix} \langle N,\ \mathbf{x}_{uu}\rangle & \langle N,\ \mathbf{x}_{vu}\rangle \\ \langle N,\ \mathbf{x}_{uv}\rangle & \langle N,\ \mathbf{x}_{vv}\rangle \end{pmatrix}. \tag{113}$$

By (112), we conclude the formula:

$$II_p(A\mathbf{x}_u + B\mathbf{x}_v) = eA^2 + 2fAB + gB^2, \quad \forall \text{ constants } A,\ B. \tag{114}$$

**Definition 2.90** *If $\mathbf{x}(u,v) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a local parametrization at $p \in S$ with basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, then the above three quantities $e,\ f,\ g$ are called* ***the coefficients of the second fundamental form*** *with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ on $T_pS$.*

**Remark 2.91** *Similar to $E,\ F,\ G$, the three quantities $e,\ f,\ g$ are* ***differentiable functions*** *on their domain. For computational purpose, it is easier to use the formula*

$$e = \langle N,\ \mathbf{x}_{uu}\rangle, \qquad f = \langle N,\ \mathbf{x}_{uv}\rangle, \qquad g = \langle N,\ \mathbf{x}_{vv}\rangle$$

*to find $e,\ f,\ g$.*

**Remark 2.92** *The first-derivative vectors $\mathbf{x}_u,\ \mathbf{x}_v,\ N_u,\ N_v$ are tangential, but their second derivatives can point to any directions in $\mathbb{R}^3$.*

**Second way:**

We have

$$
\begin{aligned}
II_p(\alpha') &= -\langle dN(\alpha'), \alpha' \rangle \\
&= -\left\langle \underbrace{(a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v)\,u' + (a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v)\,v'}, \ \mathbf{x}_u u' + \mathbf{x}_v v' \right\rangle \\
&= -(u', v') \begin{pmatrix} \langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_u \rangle & \langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_v \rangle \\ \langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_u \rangle & \langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \\
&= \boxed{-(u', v') \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix},}
\end{aligned}
$$

where $\{E, F, G\}$ are the coefficients of the first fundamental form. Thus, by comparison, the coefficients $e,\ f,\ g$ of the second fundamental form and the coefficients $E,\ F,\ G$ of the first fundamental form are related by

$$
\begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \tag{115}
$$

or equivalently,

$$
\boxed{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}}. \tag{116}
$$

We call (116) **the equations of Weingarten**.

By

$$
\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{\triangle} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}, \quad \triangle = EG - F^2 > 0.
$$

we can express $a_{ij},\ 1 \leq i,\ j \leq 2$, in (116) explicitly as

$$
a_{11} = \frac{fF - eG}{\triangle}, \quad a_{12} = \frac{gF - fG}{\triangle}, \quad a_{21} = \frac{eF - fE}{\triangle}, \quad a_{22} = \frac{fF - gE}{\triangle}. \tag{117}
$$

**Remark 2.93** *By (117), we see that $a_{ij},\ 1 \leq i,\ j \leq 2$, are **differentiable functions** on their domain.*

**Remark 2.94** *Note that in (116) both matrices on the right hand side are symmetric. But the product of two symmetric matrices are not symmetric in general.*

**Remark 2.95** *We always have $EG - F^2 > 0$, but we **do not** have $eg - f^2 > 0$ in general.*

Since the determinant of $-dN_p$ (same as the determinant of $dN_p$) is the **Gauss curvature** $K$ of $S$ at $p$, we immediately have

$$
K = k_1 k_2 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{eg - f^2}{EG - F^2}. \tag{118}
$$

Note that, in general, $eg - f^2$ may not be positive. We also have the **mean curvature** $H$ of $S$ at $p$, given by

$$
H = \frac{1}{2} Tr\left(-dN_p\right) = -\frac{1}{2}\left(a_{11} + a_{22}\right) = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}. \tag{119}
$$

**Lemma 2.96** *One can also use $e,\ f,\ g$ to classify the following: Let $p \in S$. We have:*

1. *$p$ is an **elliptic** point if $eg - f^2 > 0$.*

*2. $p$ is a **hyperbolic** point if $eg - f^2 < 0$.*

*3. $p$ is a **parabolic** point if $eg - f^2 = 0$, but at least one of $e,\ f,\ g$ is not zero.*

*4. $p$ is a **planar** point if $e = f = g = 0$ is zero.*

**Proof.** 1 and 2 are clear. For 3 and 4, if $dN_p = 0$, we have $N_u = N_v = 0$, which implies $e = f = g = 0$ due to (113). Conversely, if $e = f = g = 0$, then by (116) we have $a_{ij} = 0$ for all $1 \leq i,\ j \leq 2$, which implies $dN_p = 0$. From this observation, 3 and 4 are clear. □

**Lemma 2.97** *The two **principal curvatures** $k_1,\ k_2$ at $p \in S$ are given by*

$$k_1 = H + \sqrt{H^2 - K}, \quad k_1 = H - \sqrt{H^2 - K}.$$

*where $K$ is the **Gauss curvature** of $S$ at $p$ and $H$ is the **mean curvature** of $S$ at $p$. In particular, we see that $k_1,\ k_2$ are continuous functions on their domain and differentiable except at umbilical points (where $H^2 = K$).*

**Proof.** Since $k_1,\ k_2$ are eigenvalues of $-dN_p$, we have

$$\det\left(-dN_p - k_i I\right) = \det\left(dN_p + k_i I\right) = 0, \quad i = 1,\ 2,$$

i.e. $k_1,\ k_2$ are the two roots of the characteristic polynomial

$$\det\left(-dN_p - \lambda I\right) = \det\begin{pmatrix} a_{11} + \lambda & a_{21} \\ a_{12} & a_{22} + \lambda \end{pmatrix} = \lambda^2 - 2H\lambda + K = 0.$$

The result follows. □

**Example 2.98** *Consider the parametrization of the **torus** given by ($a > r > 0$)*

$$\mathbf{x}(u, v) = \left((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u\right), \quad 0 < u < 2\pi, \quad 0 < v < 2\pi.$$

*We compute*

$$\begin{cases} \mathbf{x}_u = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u) \\ \mathbf{x}_v = (-(a + r\cos u)\sin v,\ (a + r\cos u)\cos v,\ 0) \\ \mathbf{x}_{uu} = (-r\cos u\cos v,\ -r\cos u\sin v,\ -r\sin u) \\ \mathbf{x}_{uv} = (r\sin u\sin v,\ -r\sin u\cos v,\ 0) \\ \mathbf{x}_{vv} = (-(a + r\cos u)\cos v,\ -(a + r\cos u)\sin v,\ 0) \end{cases}$$

*and get*

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = r^2, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (a + r\cos u)^2.$$

*Also we have*

$$\begin{aligned} e = \langle N,\ \mathbf{x}_{uu} \rangle &= \frac{1}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \langle \mathbf{x}_u \wedge \mathbf{x}_v,\ \mathbf{x}_{uu} \rangle \\ &= \frac{1}{\sqrt{EG - F^2}} \det\left(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu}\right) = \frac{r^2(a + r\cos u)}{r(a + r\cos u)} = r. \end{aligned}$$

*Similarly, we obtain*

$$f = \frac{\det\left(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv}\right)}{r(a + r\cos u)} = 0, \qquad g = \frac{\det\left(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv}\right)}{r(a + r\cos u)} = (a + r\cos u)\cos u.$$

*Hence the Gauss curvature $K(u,v)$ at the point $\mathbf{x}(u,v)$ is given by*

$$K(u,v) = \frac{eg - f^2}{EG - F^2} = \frac{\cos u}{r(a + r\cos u)}, \quad 0 < u < 2\pi. \tag{120}$$

*By (120), we can easily locate elliptic points, hyperbolic points and parabolic points on the torus. See the picture in p. 160 of the book.*

**Proposition 2.99** ***(This is Proposition 1 in p. 160.)*** *Let $p \in S$ be an* ***elliptic*** *point. Then there exists a neighborhood $V$ of $p$ in $S$ such that all points in $V$ belong to the same side of $T_pS$. If $p \in S$ is a* ***hyperbolic*** *point, then in each neighborhood of $p$ there exist points of $S$ in both sides of $T_pS$.*

**Remark 2.100** ***(Important.)*** *There is no result similar to the above proposition for a parabolic or planar point.*

**Proof.** We use Taylor series expansion. Let $\mathbf{x}(u,v)$ be a parametrization near $p$ with $\mathbf{x}(0,0) = p$. The distance from the point $\mathbf{x}(u,v)$ to the tangent plane $T_pS$ is given by

$$d(u,v) = \langle \mathbf{x}(u,v) - \mathbf{x}(0,0),\ N(p) \rangle .$$

By Taylor series expansion (for vector-valued functions), we have

$$\mathbf{x}(u,v) = \mathbf{x}(0,0) + \mathbf{x}_u u + \mathbf{x}_v v + \frac{1}{2}\left(\mathbf{x}_{uu}u^2 + 2\mathbf{x}_{uv}uv + \mathbf{x}_{vv}v^2\right) + \bar{R},$$

where the derivatives are evaluated at $(0,0)$ and $\bar{R} = \bar{R}(u,v)$ satisfies (note that $\bar{R}$ is a vector)

$$\lim_{(u,v)\to(0,0)} \frac{\bar{R}}{u^2+v^2} = 0.$$

Now we can plug the above into $d(u,v)$ to get

$$\begin{aligned} d(u,v) &= \left\langle \mathbf{x}_u u + \mathbf{x}_v v + \underbrace{\frac{1}{2}\left(\mathbf{x}_{uu}u^2 + 2\mathbf{x}_{uv}uv + \mathbf{x}_{vv}v^2\right)} + \bar{R},\ N(p) \right\rangle \\ &= \underbrace{\frac{1}{2}\left(eu^2 + 2fuv + gv^2\right)} + R = \underbrace{\frac{1}{2}II_p(w)} + R \end{aligned} \tag{121}$$

where $w = \mathbf{x}_u u + \mathbf{x}_v v \in T_pS$ and

$$II_p(w) = -\langle dN_p(\mathbf{x}_u u + \mathbf{x}_v v), \mathbf{x}_u u + \mathbf{x}_v v \rangle = \frac{1}{2}\left(eu^2 + 2fuv + gv^2\right)$$

and $R = \langle \bar{R},\ N(p) \rangle$, with

$$\lim_{(u,v)\to(0,0)} \frac{R}{u^2+v^2} = 0. \tag{122}$$

We can rewrite (121) as (we may assume $w \neq 0$)

$$\boxed{d(u,v) = |w|^2\left(\frac{1}{2}II_p\left(\frac{w}{|w|}\right) + \frac{R}{|w|^2}\right)}, \quad \frac{w}{|w|} \text{ is } \textbf{unit} \text{ vector} \tag{123}$$

and note that

$$II_p\left(\frac{w}{|w|}\right) = \textbf{normal curvature} \text{ along direction } \frac{w}{|w|} \in T_pS.$$

Moreover, we also have (see (122) and Remark 2.101 below)

$$\lim_{(u,v)\to(0,0)}\frac{R}{|w|^2}=\lim_{(u,v)\to(0,0)}\frac{R}{|\mathbf{x}_u u+\mathbf{x}_v v|^2}$$
$$=\lim_{(u,v)\to(0,0)}\left(\frac{u^2+v^2}{|\mathbf{x}_u u+\mathbf{x}_v v|^2}\frac{R}{u^2+v^2}\right)=0. \tag{124}$$

If $p$ is an elliptic point (assume that $k_1(p)\geq k_2(p)>0$), then

$$II_p\left(\frac{w}{|w|}\right)\geq k_2(p)>0 \quad \text{for any nonzero } w\in T_pS.$$

Hence for $|w|\neq 0$ small enough, we have

$$\frac{1}{2}II_p\left(\frac{w}{|w|}\right)+\frac{R}{|w|^2}\geq k_2(p)+\frac{R}{|w|^2}>0, \tag{125}$$

i.e., for all $(u,v)$ close to $(0,0)$, we have $d(u,v)>0$. Thus all such $\mathbf{x}(u,v)$ lies on the same side of $T_pS$.

If $p$ is a hyperbolic point with $k_1(p)>0$ and $k_2(p)<0$), then there exist $(u,v)$ and $(\bar{u},\bar{v})$ such that $d(u,v)>0$ and $d(\bar{u},\bar{v})<0$. Hence $(u,v)$ and $(\bar{u},\bar{v})$ lie on different sides of $T_pS$. The proof is done. □

**Remark 2.101** *This is to explain that the quantity* $(u^2+v^2)/|\mathbf{x}_u u+\mathbf{x}_v v|^2$ *stays* ***bounded*** *as* $(u,v)\to(0,0)$. *We have*

$$\frac{u^2+v^2}{|\mathbf{x}_u u+\mathbf{x}_v v|^2}=\frac{u^2+v^2}{u^2|\mathbf{x}_u(u,v)|^2+2uv\langle\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)\rangle+v^2|\mathbf{x}_v(u,v)|^2} \tag{126}$$

*and as* $(u,v)\to(0,0)$, *we have*

$$\begin{cases}|\mathbf{x}_u(u,v)|^2\to|\mathbf{x}_u(0,0)|^2, \qquad |\mathbf{x}_v(u,v)|^2\to|\mathbf{x}_v(0,0)|^2\\ \langle\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)\rangle\to\langle\mathbf{x}_u(0,0),\mathbf{x}_v(0,0)\rangle.\end{cases}$$

*If we write* $(u,v)$ *as* $(r\cos\theta,r\sin\theta)$, *we get*

$$\frac{u^2+v^2}{u^2|\mathbf{x}_u(u,v)|^2+2uv\langle\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)\rangle+v^2|\mathbf{x}_v(u,v)|^2}$$
$$=\frac{r^2}{(r^2\cos^2\theta)|\mathbf{x}_u(*)|^2+(2r^2\cos\theta\sin\theta)\langle\mathbf{x}_u(*),\mathbf{x}_v(*)\rangle+\left(r^2\sin^2\theta\right)|\mathbf{x}_v(*)|^2}$$
$$=\frac{1}{(\cos^2\theta)|\mathbf{x}_u(*)|^2+(2\cos\theta\sin\theta)\langle\mathbf{x}_u(*),\mathbf{x}_v(*)\rangle+\left(\sin^2\theta\right)|\mathbf{x}_v(*)|^2}, \tag{127}$$

*where* $(*)=(r\cos\theta,r\sin\theta)$. *As* $(u,v)\to(0,0)$, *the denominator in (127) is like (the angle* $\theta$ *may not have a limit as* $(u,v)\to(0,0)$, *it will move around in the interval* $[0,2\pi]$*)*

$$(\cos\theta,\sin\theta)\begin{pmatrix}|\mathbf{x}_u(0,0)|^2 & \langle\mathbf{x}_u(0,0),\mathbf{x}_v(0,0)\rangle\\ \langle\mathbf{x}_u(0,0),\mathbf{x}_v(0,0)\rangle & |\mathbf{x}_v(0,0)|^2\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}. \tag{128}$$

*Note that the symmetric matrix*

$$A:=\begin{pmatrix}|\mathbf{x}_u(0,0)|^2 & \langle\mathbf{x}_u(0,0),\mathbf{x}_v(0,0)\rangle\\ \langle\mathbf{x}_u(0,0),\mathbf{x}_v(0,0)\rangle & |\mathbf{x}_v(0,0)|^2\end{pmatrix}$$

*is **positive-definite** and so as $\theta$ runs over $[0, 2\pi]$, the minimal quantity in (128) is equal to*

$$\min_{v\in\mathbb{R}^2,|v|=1} \langle Av, v\rangle = \lambda_2 > 0, \tag{129}$$

*where $\lambda_2 > 0$ is the **minimum eigenvalue** of $A$. Hence we have*

$$(\cos\theta, \sin\theta)\begin{pmatrix} |\mathbf{x}_u(0,0)|^2 & \langle \mathbf{x}_u(0,0), \mathbf{x}_v(0,0)\rangle \\ \langle \mathbf{x}_u(0,0), \mathbf{x}_v(0,0)\rangle & |\mathbf{x}_v(0,0)|^2 \end{pmatrix}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \geq \lambda_2 > 0 \tag{130}$$

*for all $\theta \in [0, 2\pi]$. This will imply the assertion.*

At a **parabolic** or **planar** point, there is no result similar to the above proposition. For simplicity, we only look at the case of a planar point. We can compare the following two examples.

**Example 2.102** ***(This is Example 6 in p. 145.)*** ***(Read this example by yourself.)*** *Consider the surface of revolution $S$ obtained by rotaing the curve $z = y^4$ about the $z$-axis. At the point $p = (0,0,0)$ we have $dN_p = 0$. This is because each **normal section** of $S$ at $p$ has **zero** curvature (since the curve $z = y^4$ has zero curvature at $(0,0)$). Therefore, along any unit vector direction $v \in T_pS$, the normal curvature is $\langle -dN_p(v), v\rangle = 0$. By Lemma 2.17, we have $dN_p = 0$ and so $p$ is a planar point. For this example, the whole surface $S$ lies on one side of $T_pS$.*

**Example 2.103** ***(This is Example 2 in p. 161.)*** ***(Read this example by yourself.)*** *Consider the **Monkey Saddle** parametrized by $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$, where*

$$x(u,v) = u, \quad y(u,v) = v, \quad z(u,v) = u^3 - 3v^2u.$$

*One can check that at the point $(0,0,0)$, the coefficients of the second fundamental form are $e = f = g = 0$ (due to $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = 0$ at $p$). Hence it is a planar point. However, in any neighborhood of this point, there are points on both sides of the tangent plane at $(0,0,0)$ (look at $\mathbf{x}(u,u) = (u,\ u,\ -2u^3)$, $u \in (-\infty, \infty)$). See picture in p. 162.*

### 2.2.1 The Differential Equations for the Asymptotic Curves and the Lines of Curvature.

**The equation for asymptotic curves.** Let $\mathbf{x}(u,v)$ be a parametrization near $p = \mathbf{x}(0,0) \in S$ and let $e = e(u,v)$, $f = f(u,v)$, $g = g(u,v)$ be the coefficients of the second fundamental form. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$, $t \in I$, be a regular curve on $S$ which is an asymptotic curve. Then we have

$$II_{\alpha(t)}(\alpha'(t)) = 0 \quad \text{for all} \quad t \in I. \tag{131}$$

We recall that

$$\begin{aligned} II_{\alpha(t)}(\alpha'(t)) &= -\left\langle dN_{\alpha(t)}(\alpha'(t)), \alpha'(t)\right\rangle \\ &= -\langle N_u u' + N_v v',\ \mathbf{x}_u u' + \mathbf{x}_v v'\rangle = e(u')^2 + 2fu'v' + g(v')^2. \end{aligned} \tag{132}$$

Therefore, in local coordinates, the **differential equation for an asymptotic curve** $\mathbf{x}(u(t), v(t))$ is given by

$$e(u')^2 + 2fu'v' + g(v')^2 = 0, \quad t \in I. \tag{133}$$

Note that (133) is equivalent to (131).

**Remark 2.104** *Be careful that the identity*

$$\left\langle dN_{\alpha(t)}(\alpha'(t)), \alpha'(t)\right\rangle = 0$$

*does not imply that $dN_{\alpha(t)}(\alpha'(t)) = 0$. For example, we have the following*

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 3\end{pmatrix}, \quad \left\langle \begin{pmatrix} 1 & 2 \\ 2 & 3\end{pmatrix}\begin{pmatrix} 3 \\ -1\end{pmatrix}, \begin{pmatrix} 3 \\ -1\end{pmatrix}\right\rangle = 0.$$

**Remark 2.105** *(Important.) The equation (133) is **independent of reparametrization** of $\alpha(t)$. Therefore, (136) is valid regardless of whether $\alpha(t)$ is parametrized by arc length $s$ or not.*

In particular, if $p \in S$ is a **hyperbolic point** (i.e. $eg - f^2 < 0 < 0$ at $p$), then Lemma 2.64 says that there are **two** asymptotic directions at $p$. By continuity we know $eg - f^2 < 0$ in some neighborhood of $p$ and all points on this neighborhood are hyperbolic and have two asymptotic directions. In such a case, we have the following interesting fact :

**Lemma 2.106** *Assume all points on some neighborhood $V$ around $p \in S$ are **hyperbolic** point. A necessary and sufficient condition for the **coordinate curves** ($u = u_0,\ v = v(t)$ or $u = u(t)$, $v = v_0$) near $p$ to be **asymptotic curves** is $e = g = 0$ in that neighborhood.*

**Proof.** ($\Longrightarrow$) If in some neighborhood $V$ of $p$ the two family of coordinate curves $u = u_0$, $v = v(t)$ and $u = u(t)$, $v = v_0$. are all asymptotic curves, then for the first case we have $u' \equiv 0$ and the differential equation (133) is satisfied. Therefore, we obtain

$$e(u')^2 + 2fu'v' + g(v')^2 = g(v')^2 = 0, \quad t \in I,$$

which implies $g = 0$ on $V$. Similarly for the second case we have $e = 0$ on $V$.

($\Longleftarrow$) If we have $e = g = 0$ on $V$, then equation (133) becomes $2fu'v' = 0$, where we know $f \neq 0$ everywhere on $V$ (since all points of $V$ are hyperbolic). We see that any coordinate curve in $V$ can satisfy the equation $2fu'v' = 0$. The proof is done. □

**Remark 2.107** *The above lemma says that if we can find $\mathbf{x}(u, v)$ so that $e(u, v) = g(u, v) = 0$ in a neighborhood near $p$, then **all coordinate curves** of $\mathbf{x}(u, v)$ in that neighborhood are asymptotic curves.*

**Remark 2.108** *Note that if $eg - f^2 > 0$ at $p$ (elliptic point), there is **no asymptotic curve** near $p$.*

What happens for a **parabolic** point $p \in S$, i.e. $eg - f^2 = 0$ at $p$ (but not $e = f = g = 0$ at $p$). Unlike elliptic or hyperbolic point, a parabolic point can be **isolated**. In such a case, there is not much to discuss at all. However, if we have $eg - f^2 \equiv 0$ on some neighborhood around $p$, we have:

**Lemma 2.109** *Assume all points on some neighborhood $V$ around $p \in S$ are **parabolic** points (for example, a cylinder). A necessary and sufficient condition for the **coordinate curves** $u = u_0$, $v = v(t)$, $t \in I$, lying inside $V$ to be **asymptotic curves** is $e \neq 0$ (everywhere on $V$) and $g = f = 0$ (everywhere on $V$).*

**Remark 2.110** *If the coordinate curves have the form $u = u(t)$, $v = v_0$, $t \in I$, then the condition becomes $g \neq 0$ (everywhere on $V$) and $e = f = 0$ (everywhere on $V$).*

**Proof.** By Lemma 2.64, we know there is **exactly one asymptotic direction** at each point of $V$. Assume the coordinate curve $u = u_0$, $v = v(t)$, $t \in I$, is an **asymptotic curve** lying inside $V$, then by (133) we have (note that $v'(t) \neq 0$ everywhere, for simplicity,we may assume $v(t) = t,\ t \in I$)

$$g(u_0, v(t))(v'(t))^2 = 0, \quad t \in I,$$

which gives $g(u_0, v(t)) \equiv 0$ for all $t \in I$ and so $g = 0$ everywhere on $V$. Since we also have $eg - f^2 = 0$ everywhere on $V$, we must have $f = 0$ everywhere on $V$. Moreover, since all points on $V$ are parabolic points, we must have $e \neq 0$ everywhere on $V$.

Conversely, if we have $e \neq 0$, $g = f = 0$ everywhere on $V$, then the equation (133) becomes

$$e(u')^2 + 2fu'v' + g(v')^2 = e(u')^2 = 0. \tag{134}$$

Therefore, any coordinate curve of the form $u = u_0$, $v = v(t)$, $t \in I$, is an **asymptotic curve** as long as it lies inside $V$. □

**The equation for lines of curvature.** We now turn to the differential equation for a **line of curvature**. For a line of curvature we have the condition

$$dN_{\alpha(t)}(\alpha'(t)) = \lambda(t)\,\alpha'(t), \quad \text{for all } t \in I, \quad \text{for some function } \lambda(t),$$

where $\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'$ and by

$$dN\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}$$

we have

$$dN_{\alpha(t)}(\alpha'(t)) = (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v.$$

we conclude that $u'(t)$ and $v'(t)$ satisfy the $2\times 2$ system of equations

$$\begin{cases} \dfrac{fF-eG}{EG-F^2}u' + \dfrac{gF-fG}{EG-F^2}v' = \lambda u' \cdots \ (1) \\ \dfrac{eF-fE}{EG-F^2}u' + \dfrac{fF-gE}{EG-F^2}v' = \lambda v' \cdots \ (2) \end{cases} \tag{135}$$

due to the identity

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} -G & F \\ F & -E \end{pmatrix}.$$

The system (135) is **not self-contained (which means we cannot solve it directly)**, so we need to **eliminate** the function $\lambda$. We consider $(1)\cdot v' - (2)\cdot u'$ and get

$$(fE-eF)(u')^2 + (gE-eG)u'v' + (gF-fG)(v')^2 = 0, \tag{136}$$

which may also be written as as

$$\begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0. \tag{137}$$

We call (136) the **differential equation for a line of curvature**. Note that in (136), the coefficients $E,\ F,\ G,\ e,\ f,\ g$ are all functions of $(u, v)$.

**Remark 2.111** *(**Important.**) The equation (136) is **independent of reparametrization** of $\alpha(t)$. Therefore, (136) is valid regardless of whether $\alpha(t)$ is parametrized by arc length $s$ or not.*

We note the following:

**Lemma 2.112** *(**Read this lemma by yourself.**) The $2\times 2$ system of equations (135) and the single equation (136) are equivalent.*

**Proof.** If we have equation (136), it clearly imply the system (135).

Conversely, if (136) is satisfied, then we can infer

$$\left(\frac{fF-eG}{EG-F^2}u' + \frac{gF-fG}{EG-F^2}v'\right)v' = \left(\frac{eF-fE}{EG-F^2}u' + \frac{fF-gE}{EG-F^2}v'\right)u'$$

and if $u'(t) \neq 0$ and $v'(t) \neq 0$, we can write the above as

$$\frac{1}{u'}\left(\frac{fF-eG}{EG-F^2}u' + \frac{gF-fG}{EG-F^2}v'\right) = \frac{1}{v'}\left(\frac{eF-fE}{EG-F^2}u' + \frac{fF-gE}{EG-F^2}v'\right) := \lambda(t),$$

which gives the system (135) if we use the above function as $\lambda(t)$. If $u'(t) \neq 0$ and $v'(t) = 0$, then (136) implies $(fE - eF)(u(t), v(t)) = 0$ and we can deduce the system (135) again using

$$\lambda(t) = \frac{fF - eG}{EG - F^2}(u(t), v(t)).$$

Similarly, for $u'(t) = 0$ and $v'(t) \neq 0$, then (136) implies $(gF - fG)(u(t), v(t)) = 0$ and we can deduce the system (135) again using

$$\lambda(t) = \frac{fF - gE}{EG - F^2}(u(t), v(t)).$$

Thus we conclude that (135) is **equivalent to** (136). □

We conclude:

**Lemma 2.113** *Assume all points on some neighborhood $V$ around $p$ are* ***nonumbilical*** *point. A necessary and sufficient condition for the* ***coordinate curves*** *($u = u_0,\ v = v(t)$ or $u = u(t)$, $v = v_0$) lying inside $V$ to be* ***lines of curvature*** *is $F = f = 0$ on $V$.*

**Remark 2.114** *In the above lemma, for the statement in the direction ($\Longleftarrow$) we do not have to assume that $p \in S$ is a* ***nonumbilical*** *point.*

**Proof.** For ($\Longleftarrow$), assume $\mathbf{x}(u, v)$ is a parametrization with the property $F = f \equiv 0$ on $V$. Then (136) becomes

$$(gE - eG)u'v' = 0. \tag{138}$$

Hence coordinate curves of $\mathbf{x}(u, v)$ are lines of curvature. Note that for this part we **do not** need $p$ to be a **nonumbilical** point.

For ($\Longrightarrow$), assume **coordinate curves of a parametrization $\mathbf{x}(u, v)$ are lines of curvature**, then take $u' = 0,\ v' = 1$ and $u' = 1,\ v' = 0$ respectively in (136) to get

$$gF - fG = 0 \quad \text{and} \quad fE - eF = 0, \quad \text{respectively on } V. \tag{139}$$

Since all points on $V$ are **nonumbilical** and **coordinate curves are lines of curvature (the tangent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ of coordinate curves are pointing to principal directions and the principal directions are perpendicular),** we must have $\mathbf{x}_u \perp \mathbf{x}_v$ everywhere and so $F \equiv 0$ on $V$. The above identity (139) becomes

$$fG = 0 \quad \text{and} \quad fE = 0. \tag{140}$$

Since $E > 0$ and $G > 0$ everywhere, we must have $f \equiv 0$ on $V$. The proof is done. □

### 2.2.2 Gauss and Mean Curvature for Surfaces of Revolution (Example 4 in p. 163).

Let $C$ be a **regular connected curve** lying on $xz$-plane parametrized by **arc length parameter** $v \in (a, b)$ (here we use notation $v$ instead of $s$):

$$(x, 0, z) = (\varphi(v),\ 0,\ \psi(v)), \quad v \in (a, b),$$

where $(\varphi'(v))^2 + (\psi'(v))^2 \equiv 1$ and $\varphi(v) > 0$ for all $v \in (a, b)$. Consider the surface of revolution $S$ generated by $C$ parametrized by

$$\mathbf{x}(u, v) = (\varphi(v)\cos u,\ \varphi(v)\sin u,\ \psi(v)), \quad 0 < u < 2\pi, \quad a < v < b,$$

where $\varphi(v) > 0$.

The coefficients $E,\ F,\ G$ of the first fundamental form are

$$E=\varphi^2(v)\,,\quad F=0,\quad G=(\varphi'(v))^2+(\psi'(v))^2=1,\quad \sqrt{EG-F^2}=\varphi(v)\,.$$

We compute the coefficients $e,\ f,\ g$ of the second fundamental form:

$$\begin{aligned}e&=\frac{1}{\sqrt{EG-F^2}}\det(\mathbf{x}_u,\mathbf{x}_v,\mathbf{x}_{uu})\\&=\frac{1}{\varphi(v)}\begin{vmatrix}-\varphi(v)\sin u & \varphi(v)\cos u & 0\\ \varphi'(v)\cos u & \varphi'(v)\sin u & \psi'(v)\\ -\varphi(v)\cos u & -\varphi(v)\sin u & 0\end{vmatrix}=-\varphi(v)\,\psi'(v)\,.\end{aligned}$$

Similarly, we obtain

$$f=\frac{1}{\sqrt{EG-F^2}}\det(\mathbf{x}_u,\mathbf{x}_v,\mathbf{x}_{uv})=0$$

and

$$\begin{aligned}g&=\frac{1}{\sqrt{EG-F^2}}\det(\mathbf{x}_u,\mathbf{x}_v,\mathbf{x}_{vv})\\&=\frac{1}{\varphi(v)}\begin{vmatrix}-\varphi(v)\sin u & \varphi(v)\cos u & 0\\ \varphi'(v)\cos u & \varphi'(v)\sin u & \psi'(v)\\ \varphi''(v)\cos u & \varphi''(v)\sin u & \psi''(v)\end{vmatrix}=\psi'(v)\,\varphi''(v)-\psi''(v)\,\varphi'(v)\,.\end{aligned}$$

Since $F=f=0$ (this fact remains true even if we do not use arc length parameter), we conclude that the **parallels** ($v=const.$) and the **meridians** ($u=const.$) are **lines of curvature** on $S$ (because they are coordinate curves). See Remark 2.114 also.

The Gauss curvature is

$$\begin{aligned}K&=\frac{eg-f^2}{EG-F^2}=\frac{eg}{EG}\\&=-\frac{\varphi(v)\,\psi'(v)\cdot[\psi'(v)\,\varphi''(v)-\psi''(v)\,\varphi'(v)]}{\varphi^2(v)}=-\frac{\varphi''(v)}{\varphi(v)},\end{aligned}\tag{141}$$

where, for the last identity in (141), we have used the identity

$$\frac{d}{dv}(\varphi'(v))^2+(\psi'(v))^2=2\varphi'(v)\,\varphi''(v)+2\psi'(v)\,\psi''(v)\equiv 0$$

in the numerator. In particular, we conclude: $p\in S$ is a **parabolic** point if either one of the following occurs (but not both)

$$\begin{cases}\psi'(v)=0\ (\text{same as } e=0),\\ \psi'(v)\,\varphi''(v)-\psi''(v)\,\varphi'(v)=0\ \ (\text{same as } g=0 \text{ or curvature of } C \text{ is } 0)).\end{cases}\tag{142}$$

However, if both of the above identities holds, then $p\in S$ is a **planar** point.

To find the principal curvatures, by the **equations of Weingarten** (note that $F=f=0$), we have

$$\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}=-\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}=-\begin{pmatrix}\frac{e}{E} & 0\\ 0 & \frac{g}{G}\end{pmatrix}$$

and see that the two eigenvalues $k_1,\ k_2$ of $-dN$ are

$$k_1=\frac{e}{E}=\frac{-\varphi(v)\,\psi'(v)}{\varphi^2(v)}=\frac{-\psi'(v)}{\varphi(v)}\tag{143}$$

and

$$k_2 = \frac{g}{G} = \psi'(v)\,\varphi''(v) - \psi''(v)\,\varphi'(v). \tag{144}$$

Thus

$$K = \frac{eg}{EG} = -\frac{\varphi''(v)}{\varphi(v)}$$

and

$$H = \frac{1}{2}\frac{eG + gE}{EG} = -\frac{1}{2}\frac{\psi'(v)}{\varphi(v)} + \frac{1}{2}\left[\psi'(v)\,\varphi''(v) - \psi''(v)\,\varphi'(v)\right]. \tag{145}$$

We note that $k_1,\ k_2,\ K,\ H$ are all **independent** of $u$. **They depend only on** $v$. This is intuitively obvious.

To end this example, we state one more interesting result for general surfaces:

**Lemma 2.115** *If $\mathbf{x}(u,v),\ (u,v) \in U \subset \mathbb{R}^2$, is a local parametrization of a regular surface $S \subset \mathbb{R}^3$ (not necessarily a surface of revolution) with the property*

$$F(u,v) = f(u,v) = 0, \quad \forall\ (u,v) \in U, \tag{146}$$

*then we must have*

$$k_1 = \frac{e}{E}, \quad k_2 = \frac{g}{G} \quad on \quad U. \tag{147}$$

**Proof.** If $\mathbf{x}(u,v)$ satisfies (146), the **equations of Weingarten** becomes

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = -\begin{pmatrix} \frac{e}{E} & 0 \\ 0 & \frac{g}{G} \end{pmatrix},$$

which means the $-dN$ has the two eigenvalues $\frac{e}{E}$ and $\frac{g}{G}$ and they are $k_1$ and $k_2$ respectively. The proof is done. □

### 2.2.3 Gauss and Mean Curvature for Graphs (Example 5 in p. 165).

Assume that $S : z = h(x,y)$ is the graph of a differentiable function defined on some open set $U \subset \mathbb{R}^2$. Clearly we can parametrize $S$ by

$$\mathbf{x}(x,y) = (x,\ y,\ h(x,y)), \quad (x,y) \in U$$

and get

$$\begin{cases} \mathbf{x}_x = (1,\ 0,\ h_x), \quad \mathbf{x}_y = (0,\ 1,\ h_y), \\ \mathbf{x}_{xx} = (0,\ 0,\ h_{xx}), \quad \mathbf{x}_{xy} = (0,\ 0,\ h_{xy}), \quad \mathbf{x}_{yy} = (0,\ 0,\ h_{yy}), \\ E(x,y) = 1 + h_x^2, \quad F(x,y) = h_x h_y, \quad G(x,y) = 1 + h_y^2, \\ EG - F^2 = 1 + h_x^2 + h_y^2, \end{cases}$$

and then

$$N(x,y) = \frac{(-h_x,\ -h_y,\ 1)}{\left(1 + h_x^2 + h_y^2\right)^{1/2}}$$

and

$$\begin{cases} e = \langle N,\ \mathbf{x}_{xx} \rangle = \dfrac{h_{xx}}{\left(1 + h_x^2 + h_y^2\right)^{1/2}}, \\ f = \langle N,\ \mathbf{x}_{xy} \rangle = \dfrac{h_{xy}}{\left(1 + h_x^2 + h_y^2\right)^{1/2}}, \\ g = \langle N,\ \mathbf{x}_{yy} \rangle = \dfrac{h_{yy}}{\left(1 + h_x^2 + h_y^2\right)^{1/2}}. \end{cases} \tag{148}$$

We conclude:

**Lemma 2.116** *The Gauss curvature and mean curvature of a graphic surface* $z = h(x,y)$, $(x,y) \in U$, *are given by*

$$K(x,y) = \frac{eg - f^2}{EG - F^2} = \frac{h_{xx}h_{yy} - h_{xy}^2}{\left(1 + h_x^2 + h_y^2\right)^2}, \quad (x,y) \in U \tag{149}$$

*and*

$$\begin{aligned} H(x,y) &= \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} \\ &= \frac{1}{2}\frac{\left(1 + h_y^2\right)h_{xx} - 2h_x h_y + \left(1 + h_x^2\right)h_{yy}}{\left(1 + h_x^2 + h_y^2\right)^{3/2}}, \quad (x,y) \in U. \end{aligned} \tag{150}$$

### 2.2.4 Gauss and Mean Curvature for Graphs with Special Coordinates (continue Example 5 in p. 165).

At a point $p \in S$, it is possible to choose a coordinate system so that $T_pS$ is the $xy$-plane, $N$ is pointing in the positive $z$ direction, $p$ is the origin $(0,0,0)$, and near $p$, the surface $S$ is the graph of a function $z = h(x,y)$, $(x,y) \in U$, where $h$ satisfies

$$h(0,0) = 0, \quad h_x(0,0) = 0, \quad h_y(0,0) = 0. \tag{151}$$

One can see Exercise 26 in p. 93 for the above properties.

Now if we use the above-mentioned parametrization $\mathbf{x}(x,y) = (x, y, h(x,y))$, $(x,y) \in U$, we have

$$E(0,0) = 1, \quad F(0,0) = 0, \quad G(0,0) = 1 \tag{152}$$

and

$$e(0,0) = h_{xx}(0,0), \quad f(0,0) = h_{xy}(0,0), \quad g(0,0) = h_{yy}(0,0). \tag{153}$$

By **the equations of Weingarten**, we have

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} &= -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \\ &= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\begin{pmatrix} h_{xx}(0,0) & h_{xy}(0,0) \\ h_{xy}(0,0) & h_{yy}(0,0) \end{pmatrix}. \end{aligned} \tag{154}$$

So the map $dN_p : T_pS \to T_pS$ with respect to the **orthonormal basis** $\{\mathbf{x}_x, \mathbf{x}_y\} = \{(1,0), (0,1)\}$ of the $xy$-plane is given by

$$-dN_p\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h_{xx}(0,0) & h_{xy}(0,0) \\ h_{xy}(0,0) & h_{yy}(0,0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{155}$$

and the **second fundamental form** $II_p(v) = -\langle dN_p(v), v\rangle : T_pS \to \mathbb{R}$ is given by

$$II_p(v) = h_{xx}(0,0)\,x^2 + 2h_{xy}(0,0)\,xy + h_{yy}(0,0)\,y^2, \quad v = (x,y). \tag{156}$$

The matrix in (155) is known as the **Hessian matrix** of $h$ at $(0,0)$. Note that at this moment we may not have $h_{xy}(0,0) = 0$. However, we can make $h_{xy}(0,0) = 0$ by a rotation in the $xy$-plane.

If $k_1 = k_2$ (denote the common number as $k$) at $p$, then we must have $-dN_p = kI$ and we have

$$h_{xx}(0,0) = k, \quad h_{xy}(0,0) = 0, \quad h_{yy}(0,0) = k$$

for the above **orthonormal basis** $\{\mathbf{x}_x, \mathbf{x}_y\} = \{(1,0), (0,1)\}$.

If $k_1 \neq k_2$, the two **principal directions** $e_1$ and $e_2$ must be perpendicular to each other and we can **rotate** the above $xy$-plane so that the $x$ and $y$ axes are directed along the **principal directions**

$e_1$ and $e_2$. That is, after rotation, we have $e_1 = (1,0)$ and $e_2 = (0,1)$. Since they are eigenvectors of $-dN_p : T_pS \to T_pS$, by (155), we have

$$-dN_p\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}h_{xx}(0,0) & h_{xy}(0,0)\\ h_{xy}(0,0) & h_{yy}(0,0)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}h_{xx}(0,0)\\ h_{xy}(0,0)\end{pmatrix} = k_1\begin{pmatrix}1\\0\end{pmatrix}$$

and

$$-dN_p\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}h_{xy}(0,0)\\ h_{yy}(0,0)\end{pmatrix} = k_2\begin{pmatrix}0\\1\end{pmatrix}$$

Hence we must have

$$h_{xx}(0,0) = k_1, \quad h_{xy}(0,0) = 0, \quad h_{yy}(0,0) = k_2. \tag{157}$$

and by (153) we conclude

$$e(0,0) = h_{xx}(0,0) = k_1, \quad f(0,0) = h_{xy}(0,0) = 0, \quad g(0,0) = h_{yy}(0,0) = k_2. \tag{158}$$

The matrix for $-dN_p$ with respect to $\{\mathbf{x}_x, \mathbf{x}_y\} = \{(1,0),(0,1)\}$ is now diagonal, given by

$$\begin{pmatrix}h_{xx}(0,0) & h_{xy}(0,0)\\ h_{xy}(0,0) & h_{yy}(0,0)\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}.$$

By Taylor series expansion, the function $z = h(x,y)$ near $(0,0)$ has the form (note that we have (151))

$$\begin{aligned} h(x,y) &= \frac{1}{2}\left(h_{xx}(0,0)x^2 + 2h_{xy}(0,0)xy + h_{yy}(0,0)y^2\right) + R(x,y) \\ &= \frac{1}{2}\left(k_1x^2 + k_2y^2\right) + R(x,y), \end{aligned} \tag{159}$$

where $R(x,y)$ satisfies

$$\lim_{(x,y)\to(0,0)} \frac{R(x,y)}{x^2+y^2} = 0.$$

### 2.2.5 Geometric Interpretation of the Gauss Curvature.

Let $S$ and $\tilde{S}$ be two orientable surfaces with orientation $N$ (differentiable unit normal vector field on $S$) and $\tilde{N}$ (differentiable unit normal vector field on $\tilde{S}$) respectively. Let $\varphi : S \to \tilde{S}$ be a differentiable map and assume at $p \in S$ the map $d\varphi_p : T_pS \to T_{\varphi(p)}\tilde{S}$ is nonsingular.

**Definition 2.117** *We say the map $\varphi : S \to \tilde{S}$ is **orientation-preserving** at $p$ if, for any **positive** basis $\{v,w\}$ on $T_pS$ (which means $v \wedge w$ is pointing in the direction of $N(p)$, i.e. $\det(v,w,N(p)) > 0$), the basis $\{d\varphi_p(v), d\varphi_p(w)\}$ on $T_{\varphi(p)}\tilde{S}$ is also **positive** on $T_{\varphi(p)}\tilde{S}$ (which means $d\varphi_p(v) \wedge d\varphi_p(w)$ is pointing in the direction of $\tilde{N}(\varphi(p))$, i.e. $\det\left(d\varphi_p(v), d\varphi_p(w), \tilde{N}(\varphi(p))\right) > 0$)). Otherwise, the map $\varphi : S \to \tilde{S}$ is called **orientation-reversing** at $p$, which means that $d\varphi_p : T_pS \to T_{\varphi(p)}\tilde{S}$ maps some positive basis on $T_pS$ into negative basis on $T_{\varphi(p)}\tilde{S}$ $\{v,w\}$.*

Let $S$ be an orientable surface with orientation $N : S \to S^2$ and $dN_p$ is **nonsingular** at $p \in S$. By the **inverse function theorem**, there is a **small** neighborhood $V \subset S$ around $p \in S$ such that

$$N : V \subset S \to N(V) \text{ (denote it as } \tilde{V}) \subset S^2 \tag{160}$$

is a **diffeomorphism** and either $K > 0$ everywhere on $V$ or $K < 0$ everywhere on $V$.

The Gauss map $N : V \to \tilde{V}$ will **induce an orientation** on $\tilde{V} \subset S^2$ (since we can **identify** $T_{N(p)}S^2$ as $T_pS$ and choose the orientation on $T_{N(p)}S^2$ to be the **same** as the orientation on $T_pS$). By the identity

$$dN_q(v) \wedge dN_q(w) = K(q)(v \wedge w), \quad \forall\, q \in V, \quad v,\ w \in T_qS, \tag{161}$$

we see that if $K > 0$ everywhere on $V$, then both $v \wedge w$ and $dN_q(v) \wedge dN_q(w)$ are pointing to the same direction. That is, if $\{v, w\}$ is positive on $T_qS$ (the domain space of $dN_q$), then $\{dN_q(v), dN_q(w)\}$ is also positive on $T_qS$ (the target space of $dN_q$). By definition, the Gauss map $N : V \to \tilde{V}$ is **orientation-preserving** at any $q \in V$. On the other hand, if $K < 0$ everywhere on $V$, the Gauss map $N : V \to \tilde{V}$ is **orientation-reversing** at any $q \in V$. Therefore, we conclude:

**Lemma 2.118** *$N : V \to \tilde{V}$ is orientation-preserving (orientation-reversing) at all $q \in V$ if and only if $K > 0$ $(K < 0)$ everywhere on $V$.*

**Remark 2.119** *Let $\alpha(s) \in V$ (on $V$ there is an orientation $N$), $s \in I$, be a small simple closed curve enclosing $p$ in its interior and is **counterclockwise,** which means that when you walk along $\alpha(s) \in V$ in the length-increasing direction, the vector $\alpha'(s_0) \wedge \alpha'(s_0 + \varepsilon)$ ($\varepsilon > 0$ is small) is pointing to the direction of $N(\alpha(s_0))$ for all $s_0 \in I$. Now by (161) we have*

$$\begin{aligned} & \left.\frac{d}{ds}\right|_{s=s_0} N(\alpha(s)) \wedge \left.\frac{d}{ds}\right|_{s=s_0+\varepsilon} N(\alpha(s)), \qquad N(\alpha(s)) \in S^2 \\ &= \left[dN_{\alpha(s_0)}(\alpha'(s_0)) \wedge dN_{\alpha(s_0+\varepsilon)}(\alpha'(s_0+\varepsilon))\right] \approx K(\alpha(s_0)) \left[\alpha'(s_0) \wedge \alpha'(s_0+\varepsilon)\right]. \end{aligned} \tag{162}$$

*If we have $K > 0$ on $V$, then the vector $dN_{\alpha(s_0)}(\alpha'(s_0)) \wedge dN_{\alpha(s_0+\varepsilon)}(\alpha'(s_0+\varepsilon))$ is also pointing to the direction of $N(\alpha(s_0))$ (note that $N(\alpha(s_0)) \in S^2$ and we choose the normal $N$ on $S^2$ at $N(\alpha(s_0))$ as $N(\alpha(s_0))$, i.e. $N(N(\alpha(s_0))) = N(\alpha(s_0))$). Therefore, the curve $N(\alpha(s))$ on $S^2$ also has **counterclockwise** orientation as $s$ is increasing. On the other hand, if $K < 0$ on $V$, the curve $N(\alpha(s))$ on $S^2$ has the property that $\left.\frac{d}{ds}\right|_{s=s_0} N(\alpha(s)) \wedge \left.\frac{d}{ds}\right|_{s=s_0+\varepsilon} N(\alpha(s))$ is pointing to the direction of $-N(\alpha(s_0))$. Therefore, the curve $N(\alpha(s))$ on $S^2$ has **clockwise** orientation as $s$ is increasing.*

We use the convention that if $K > 0$ in $V$, then the area of the set $N(V)$ in $S^2$ has a **positive sign**. While if $K < 0$ in $V$, then the area of the set $N(V)$ in $S^2$ has a **negative sign**. Under this convention, $N(V)$ has a **signed area**.

With the above convention, we can state the following:

**Proposition 2.120** ***(This is Proposition 2 in p. 169.)** **(Geometric meaning of the Gauss curvature.)** Let $p \in S$ (with Gauss map $N : S \to S^2$) such that $K(p) \neq 0$, and $V$ be a connected neighborhood around $p \in S$ such that either $K > 0$ in $V$ or $K < 0$ in $V$ (here $N : V \to N(V)$ is a diffeomorphism). Then*

$$K(p) = \lim_{A \to 0} \frac{A'_{sign}}{A}, \tag{163}$$

*where $A$ is the area of the region $B \subset V$ containing $p$, $A'_{sign}$ is the **signed area** of $N(B)$ in $S^2$, and the limit is taken through a sequence of regions $B_n$ that converges to $p$ in the sense that any sphere around $p$ contains all $B_n$, for $n$ sufficiently large.*

**Remark 2.121** *Explain the meaning of $B_n$ that converges to $p$ as $n \to \infty$.*

**Proof.** The area of $B \subset S$ is given by

$$A = \iint_R |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv,$$

where $\mathbf{x}(u, v) : R \subset U \subset \mathbb{R}^2 \to S$ is a parametrization near $p \in S$ with $\mathbf{x}(R) = B$. Also the area of the region $N(B)$ in $S^2$ is given by

$$A' = \iint_R |N_u \wedge N_v| \, dudv$$

since $N(u,v) = N(\mathbf{x}(u,v)) : R \subset U \subset \mathbb{R}^2 \to S^2$ is a parametrization near $N(p) \in S^2$ with $N(R) = N(B)$. By the identity (see Lemma 2.78)

$$|N_u \wedge N_v| = |dN(\mathbf{x}_u) \wedge dN(\mathbf{x}_v)| = |K| |\mathbf{x}_u \wedge \mathbf{x}_v|,$$

we have

$$A'_{sign} = \iint_R K |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv.$$

Hence

$$\lim_{A\to 0} \frac{A'_{sign}}{A} = \lim_{R\to 0} \frac{\frac{1}{R}\iint_R K |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv}{\frac{1}{R}\iint_R |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv} = \frac{K(p) |\mathbf{x}_u \wedge \mathbf{x}_v| (p)}{|\mathbf{x}_u \wedge \mathbf{x}_v| (p)} = K(p).$$

□

**Remark 2.122** *We can compare the above lemma with the curve case. Recall that for a regular parametrized curve $C \subset \mathbb{R}^2$ we have*

$$\text{signed curvature} = k(p) = \frac{d\theta}{ds}(p) = \lim_{\triangle s\to 0} \frac{\triangle\theta}{\triangle s}, \quad \theta \text{ is the tangent angle,}$$

*where $\triangle\theta$ is the **signed length** on the unit circle $S^1$ of the image of $\triangle s$ under the map of the unit tangent vector $T$ ($\triangle\theta > 0$ for counterclockwise orientation curve and $\triangle\theta < 0$ for clockwise orientation curve). But it is the same as the signed length on $S^1$ of the image of $\triangle s$ under the map of the unit normal vector $N$.*

To Be Continued

### 2.2.6 The Hessain of a regular surface in $\mathbb{R}^3$.

@@@@@@@@@@@@@@@@@@@@@@ *From Here Next Time* @@@@@@@@@@@@@@@@@@@@@@